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Smokin' Joe's Catalog of Groups: Dihedral Groups

Given an integer $n \geq 2$, the ***dihedral group or order*** $\boldsymbol{2n}$ is denoted by D_n and has the presentation

$$D_n = \langle r, s \mid |r| = n, |s| = 2, s \notin \langle r \rangle, sr = r^{n-1}s \rangle$$

The group D_2 is abelian and is isomorphic to the Klein four-group. For $n > 2$, D_n is a nonabelian group.

Let $N = \langle r \rangle$ be the subgroup generated by r. Since $|D_n : N| = 2$, N is a normal subgroup of D_n and

$$\begin{aligned} D_n &= N \cup Ns \\ &= \{e, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\} \end{aligned}$$

Suppose n is odd; for example, the Cayley digraph of D_5 using generating set $\{r, s\}$ is shown in Figure 1.

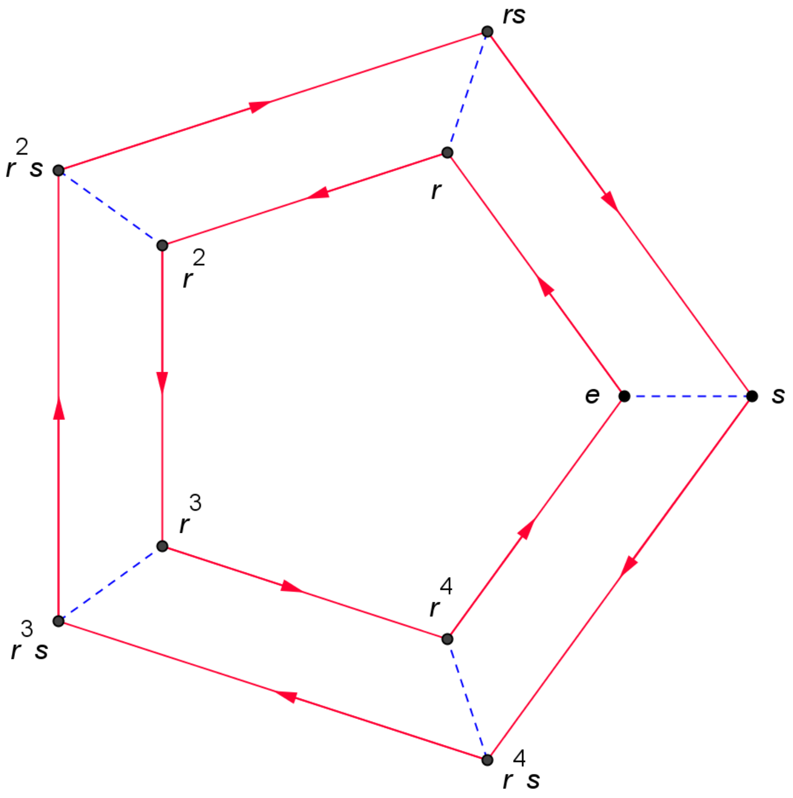

Figure 1 Cayley digraph for D_5 using generating set $\{r, s\}$

Using the Cayley digraph for D_5, it is easy to check that

$$sr^k = r^{5-k}s$$

for each k, $1 \leq k < 5$. It follows that each of elements s, rs, r^2s, r^3s, and r^4s has order 2. Likewise, in D_n, with n odd,

$$sr^k = r^{n-k}s$$

for each k, $1 \leq k < n$. Hence, each of the elements in the coset Ns has order 2, and these are the only elements of order 2. Since N, itself, is a cyclic group of order n, we have in the case when n is odd that D_n has n elements of order 2 and $\phi(m)$ elements of order m for each factor m of n. (See *Cyclic Groups*.)

It also follows that the center $\mathcal{C}$ of D_n is trivial when n is odd.

Next, suppose n is even, $n \geq 4$; for example, the Cayley digraph for D_6 using generating set $\{r, s\}$ is shown in Figure 2.

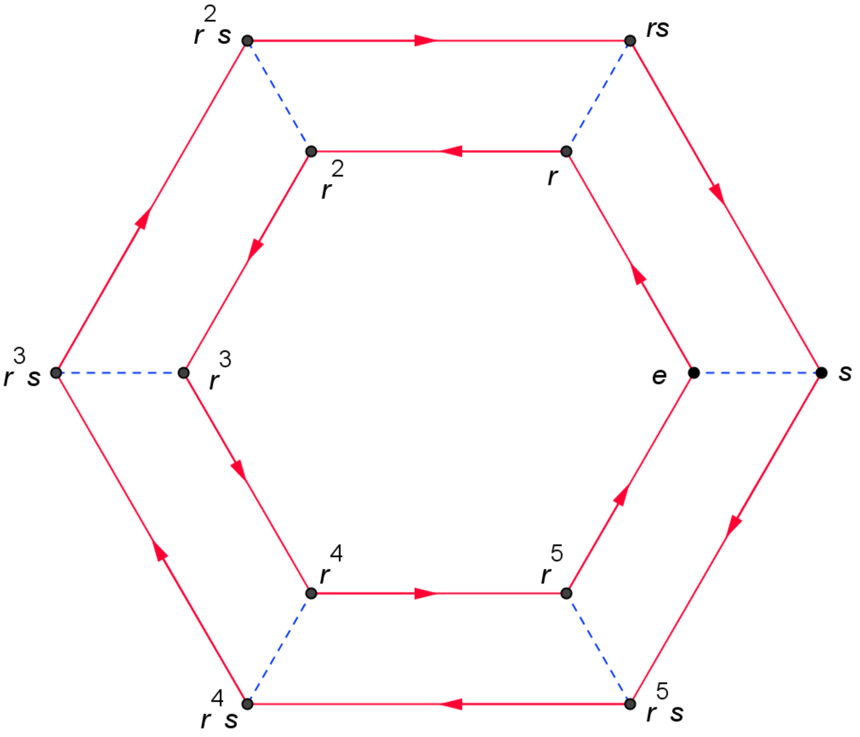

Figure 2 Cayley digraph for D_6 using generating set $\{r, s\}$

Using the Cayley digraph for D_6, it is easy to check that

$$sr^k = r^{6-k}s$$

for each k, $1 \leq k < 6$. It follows that each of elements s, rs, r^2s, r^3s, r^4s, and r^5s has order 2. Likewise, in D_n, with n even,

$$sr^k = r^{n-k}s$$

for each k, $1 \leq k < n$. Hence, each of the elements in the coset Ns has order 2. In addition, the element $r^{n/2}$ has order 2. Since N, itself, is a cyclic group of order n, we have in the case when n is even that D_n has $n+1$ elements of order 2 and $\phi(m)$ elements of order m for each factor m of n, $m > 2$. (See *Cyclic Groups*.)

Also, since $sr^{n/2} = r^{n/2}s$, the element $r^{n/2}$ commutes with s. It follows that $r^{n/2}$ belongs to the center $\mathcal{C}$ of D_n when n is even. In fact, it can be shown that $\mathcal{C} = \{e, r^{n/2}\}$ in this case.

The group D_n arises often as the group of symmetries of some mathematical object in the plane or in space.

An *isometry* of $\mathbb{R}^2$ is a bijection $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that is distance-preserving; that is, for any two points X_1, X_2 in the plane,

$$X_1X_2 = f(X_1)f(X_2)$$

where AB denotes the (Euclidean) distance between the points A and B. Given a subset $\mathcal{F}$ (a "figure") of $\mathbb{R}^2$, a *symmetry* of $\mathcal{F}$ is an isometry that maps $\mathcal{F}$ to $\mathcal{F}$.

Given a figure $\mathcal{F}$, it is clear that the identity mapping ϵ is a symmetry of $\mathcal{F}$. We leave it as an exercise to show that, if α and β are symmetries of $\mathcal{F}$, then so are α^{-1} and $\beta \circ \alpha$. It follows, by SJST (Smokin' Joe's Subgroup Test), that the set of symmetries of $\mathcal{F}$, under the operation of composition, is a group. (In fact, it is a subgroup of the group of permutations of $\mathbb{R}^2$.)

Example 1: Let n be a positive integer, $n \geq 3$, and consider a regular n-gon $\mathcal{P}_n$. Without loss of generality, we place $\mathcal{P}_n$ in $\mathbb{R}^2$ (or, the complex plane $\mathbb{C}$) so that its vertices are at the n th roots of unity.

Let's begin with the case $n = 3$, that is, with an equilateral triangle $\mathcal{P}_3$ — refer to Figure 3. We place its three vertices at the cube roots of unity:

$$A = (1,0) = 1 + 0i, \quad B = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \quad C = \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right) = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

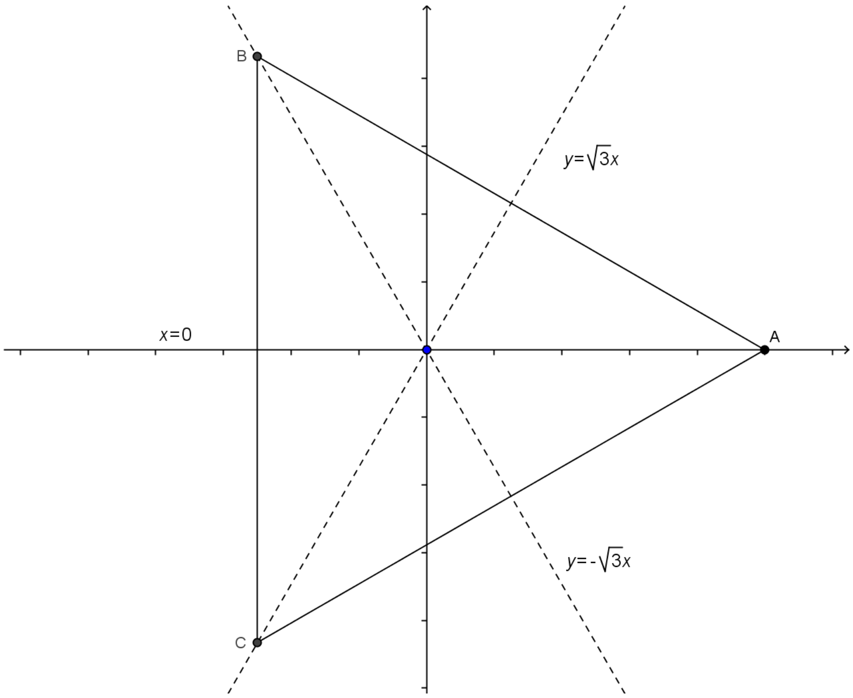

Figure 3 An equilateral triangle

(Note that the numbers $z_1 = 1$, $z_2 = (-1 + \sqrt{3}i)/2$, and $z_3 = (-1 - \sqrt{3}i)/2$ are the three solutions to the equation $z^3 = 1$; hence, the name, “cube roots of unity.”)

What are the symmetries of $\mathcal{P}_3$? Well, any symmetry f of $\mathcal{P}_3$ must map $\mathcal{P}_3$ to $\mathcal{P}_3$, and this completely determines f, since an isometry of $\mathbb{R}^2$ is completely determined by the image of any triangle. Also, since f is distance-preserving, the origin $(0, 0)$ must be a fixed point of f. It follows that f is either (1) a rotation about the origin, or (2) a reflection about some line through the origin.

Also, since any symmetry of $\mathcal{P}_3$ maps $\mathcal{P}_3$ to $\mathcal{P}_3$, it must permute the vertices A, B, and C of $\mathcal{P}_3$. Thus, any symmetry of $\mathcal{P}_3$ may be regarded as a permutation of $\{A, B, C\}$; that is, as an element of the symmetric group $\mathcal{S}(\{A, B, C\}) \cong S_3$ (see *Permutation Groups*).

From the preceding observations, we may conclude that the following are the symmetries of $\mathcal{P}_3$, with each symmetry described geometrically and as a permutation of $\{A, B, C\}$:

$$\begin{aligned}
\epsilon &= \text{identity} = (A)\\
\rho &= \text{rotation}, (0,0), 120^\circ = (A\,B\,C)\\
\rho^2 &= \text{rotation}, (0,0), 240^\circ = (A\,C\,B)
\end{aligned}$$

$$\begin{aligned} \sigma &= \text{reflection}, x\text{-axis} = (B\,C) \\ \sigma\rho &= \text{reflection}, y = \sqrt{3}\,x; \sigma\rho = (A\,B) \\ \sigma\rho^2 &= \text{reflection}, y = -\sqrt{3}\,x; \sigma\rho^2 = (A\,C) \end{aligned}$$

Note that $|\rho| = 3$, $|\sigma| = 2$, and $\sigma\rho = \rho^2\sigma$. Thus, the group of symmetries of $\mathcal{P}_3$ is isomorphic to D_3.

Next, consider the case $n = 4$, that is, a square $\mathcal{P}_4$ — refer to Figure 4. We place its four vertices at the fourth roots of unity:

$$\begin{aligned} A &= (1,0) = 1 + 0i, & B &= (0,1) = i \\ C &= (-1,0) = -1 + 0i, & D &= (0,-1) = -i \end{aligned}$$

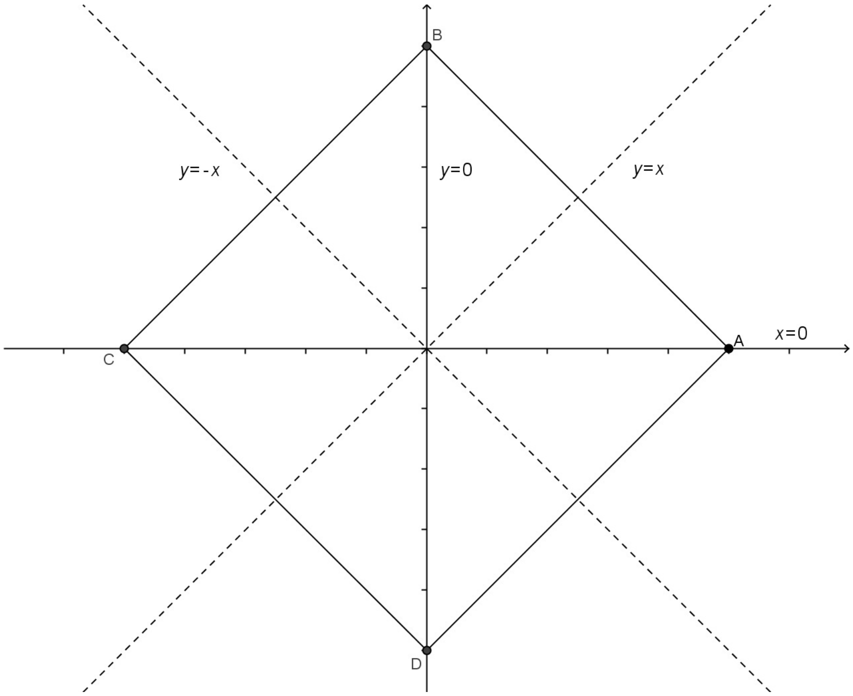

Figure 4 A square

What are the symmetries of $\mathcal{P}_4$? Well, as with $\mathcal{P}_3$, any symmetry f of $\mathcal{P}_4$ must map $\mathcal{P}_4$ to $\mathcal{P}_4$, and must fix the origin $(0,0)$. It follows that f is either (1) a rotation about the origin, or (2) a reflection about some line through the origin.

Also, as with $\mathcal{P}_3$, any symmetry of $\mathcal{P}_4$ must permute the vertices A, B, C, and D of $\mathcal{P}_4$. Thus, any symmetry of $\mathcal{P}_4$ may be regarded as a permutation of $\{A, B, C, D\}$, that is, as an element of $\mathcal{S}(\{1,2,3,4\}) \cong S_4$ (see *Permutation Groups*).

From these observations, we may conclude that the following are the symmetries of $\mathcal{P}_4$, with each symmetry described geometrically and as a permutation of $\{A, B, C, D\}$:

$$\begin{aligned}
\epsilon &= \text{identity} = (A)\\
\rho &= \text{rotation}, (0,0), 90^\circ = (A\,B\,C\,D)\\
\rho^2 &= \text{rotation}, (0,0), 180^\circ = (A\,C)(B\,D)\\
\rho^3 &= \text{rotation}, (0,0), 270^\circ = (A\,D\,C\,B)\\
\sigma &= \text{reflection}, x\text{-axis} = (B\,D)\\
\sigma\rho &= \text{reflection}, y = x; \sigma\rho = (A\,B)(C\,D)\\
\sigma\rho^2 &= \text{reflection}, y\text{-axis}; \sigma\rho^2 = (A\,C)\\
\sigma\rho^3 &= \text{reflection}, y = -x; \sigma\rho^3 = (A\,D)(B\,C)
\end{aligned}$$

Note that $|\rho| = 4$, $|\sigma| = 2$, and $\sigma\rho = \rho^3\sigma$. Thus, the group of symmetries of $\mathcal{P}_4$ is isomorphic to D_4.

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Generalizing the preceding examples, consider a regular n-gon $\mathcal{P}_n$, $n \geq 3$, placed in $\mathbb{R}^2$ so that its vertices $A_0, A_1, \ldots, A_{n-1}$ are at the n th roots of unity, with $A_0 = (1, 0)$.

If n is odd, then the group G_n of symmetries of $\mathcal{P}_n$ is generated by:

$$\begin{aligned}
\rho &= \text{rotation}, (0,0), (360/n)^\circ = (A_0\,A_1\ldots\,A_{n-1})\\
\sigma &= \text{reflection}, x\text{-axis} = (A_1\,A_{n-1})\cdots(A_{(n-1)/2}\,A_{(n+1)/2})
\end{aligned}$$

Similarly, if n is even, then the group G_n is generated by

$$\begin{aligned}
\rho &= \text{rotation}, (0,0), (360/n)^\circ = (A_0\,A_1\ldots\,A_{n-1})\\
\sigma &= \text{reflection}, x\text{-axis} = (A_1\,A_{n-1})\cdots(A_{(n-2)/2}\,A_{(n+2)/2})
\end{aligned}$$

In either case, we leave it as an exercise to show that $|\rho| = n$, $|\sigma| = 2$, and $\sigma\rho = \rho^{-1}\sigma$. It follows that $G_n \cong D_n$.

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Theorem 1: Let n be a positive integer, $n \geq 3$. The group G_n of symmetries of a regular n-gon $\mathcal{P}_n$ in a plane is isomorphic to D_n.

■

As with $\mathbb{R}^2$, an *isometry* of $\mathbb{R}^3$ is a bijection $f : \mathbb{R}^3 \to \mathbb{R}^3$ that is distance-preserving; that is, for any two points X_1 and X_2 in $\mathbb{R}^3$,

$$X_1X_2 = f(X_1)f(X_2)$$

where AB denotes the (Euclidean) distance between the points A and B. Given a subset $\mathcal{F}$ of $\mathbb{R}^3$ (a "figure"), a *symmetry* of $\mathcal{F}$ is an isometry that maps $\mathcal{F}$ to $\mathcal{F}$.

As an exercise, you are asked to show that, given a figure $\mathcal{F}$ in $\mathbb{R}^3$, the set of symmetries of $\mathcal{F}$, under the operation of composition, is a group (in fact, it is a subgroup of $(\mathcal{S}(\mathbb{R}^3),\ \circ\ )$; see *Permutation Groups*).

Next, let's work out the group of symmetries for some well-known polyhedra. A nice family to study are the regular prisms. Given an integer n, $n \geq 3$, a *regular n-prism* has two ends, both of which are regular n-gons, and n sides, each of which is a rectangle. In the case $n = 3$, the regular 3-prism is also called a *regular triangular prism*; let's find its symmetry group.

Example 2: Consider a regular triangular prism $\mathcal{R}_3$ in $\mathbb{R}^3$ with its vertices at:

$$\begin{array}{lll} A = (2,0,1), & B = (-1,\sqrt{3},1), & C = (-1,-\sqrt{3},1) \\ D = (2,0,-1), & E = (-1,\sqrt{3},-1), & F = (-1,-\sqrt{3},-1) \end{array}$$

Any symmetry of $\mathcal{R}_3$ must fix $Z = (0,0,0)$ and permute the vertices; hence, the group of symmetries of $\mathcal{R}_3$ is a subgroup of $S(\{A,B,C,D,E,F\}) \cong S_6$ (see *Permutation Groups*). Let's begin by obtaining an upper bound on the number of symmetries of $\mathcal{R}_3$.

In general, let X' denote the image of the vertex of X under an arbitrary symmetry of $\mathcal{R}_3$. Assume that A' can be any element of $\{A,B,C,D,E,F\}$. Thus, there are 6 possibilities for A'. Given A', there are 2 choices for B', since $\overline{A'B'}$ must be an edge of one end of $\mathcal{R}_3$. The choices for A' and B' determine C', and hence the symmetry, since $ZABC$ is a tetrahedron. Therefore, $\mathcal{R}_3$ has at most $6(2) = 12$ symmetries.

Let's find the rotational symmetries of $\mathcal{R}_3$. For this, we note that the axis of the prism — the z-axis in our case — is an axis of 3-fold rotational symmetry. Also, any one of the three lines determined by the center of the prism and the midpoint of one of the three lateral edges (for example, the midpoint M of $\overline{AD}$) is a axis of 2-fold rotational symmetry. This yields the following rotational symmetries:

$$\begin{aligned} &\rho_1 = \text{rotation}, 120^\circ, z\text{-axis}; \rho_1 = (A\,B\,C)(D\,E\,F) \\ &\rho_1^2 = \text{rotation}, 240^\circ, z\text{-axis}; \rho_1^2 = (A\,C\,B)(D\,F\,E) \\ &\rho_2 = \text{rotation}, 180^\circ, x\text{-axis}; \rho_2 = (A\,D)(B\,F)(C\,E) \\ &\rho_3 = \text{rotation}, 180^\circ, y = \sqrt{3}\,x, z = 0; \rho_3 = (A\,E)(B\,D)(C\,F) \\ &\rho_4 = \text{rotation}, 180^\circ, y = -\sqrt{3}\,x, z = 0; \rho_4 = (A\,F)(B\,E)(C\,D) \end{aligned}$$

Note that $\rho_1\rho_2 = \rho_4$ and $\rho_2\rho_1 = \rho_3 = \rho_1^2\rho_2$. Thus, the subgroup $\mathcal{H}$ of rotational symmetries is generated by ρ_1 and ρ_2, and $\mathcal{H} \cong D_3$.

Let σ denote reflection through the xy-plane. Then σ is a symmetry of $\mathcal{R}_3$ and $\sigma \notin \mathcal{H}$. It follows that the group $\mathcal{G}$ of symmetries of $\mathcal{R}_3$ has order 12, that $\mathcal{H} \lhd \mathcal{G}$, and that $\mathcal{G} = \mathcal{H} \cup \sigma\mathcal{H}$. Here are the remaining six (non-rotational) symmetries of $\mathcal{R}_3$:

$$\begin{aligned}
\sigma &= (A\,D)(B\,E)(C\,F) = \text{reflection}, z = 0 \\
\sigma\rho_1 &= (A\,E\,C\,D\,B\,F); (\sigma\rho_1)(x,y,z) = \Big((-x+\sqrt{3}\,y)/2, (\sqrt{3}\,x - y)/2, -z\Big) \\
\sigma\rho_1^2 &= (A\,F\,B\,D\,C\,E); (\sigma\rho_1^2)(x,y,z) = \Big((-x+\sqrt{3}\,y)/2, (-\sqrt{3}\,x - y)/2, -z\Big) \\
\sigma\rho_2 &= (B\,C)(E\,F) = \text{reflection}, y = 0 \\
\sigma\rho_3 &= (A\,B)(D\,E) = \text{reflection}, y = \sqrt{3}\,x \\
\sigma\rho_4 &= (A\,C)(D\,F) = \text{reflection}, y = -\sqrt{3}\,x
\end{aligned}$$

In summary, the group $\mathcal{G}$ of symmetries of a triangular prism $\mathcal{R}_3$ is a nonabelian group of order 12. It has two elements of order 6, two elements of order 3, and seven elements of order 2. Hence, G has the same "order profile" as does D_6.

To see that $\mathcal{G}$ is isomorphic to D_6, note that $\mathcal{G}$ is generated by $\sigma\rho_1$ and ρ_2, $\sigma\rho_1$ has order 6, ρ_2 has order 2, and

$$\rho_2(\sigma\rho_1) = \sigma\rho_3 = (\sigma\rho_1)^{-1}\rho_2$$

Hence, the mapping $\phi : \mathcal{G} \to D_6$ determined by $\phi(\sigma\rho_1) = r$ and $\phi(\rho_2) = s$ is an isomorphism.

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Given a figure $\mathcal{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, it is often useful to express a symmetry f of $\mathcal{F}$ using function notation; that is, by giving $f(x,y)$ or $f(x,y,z)$ as a formula. The figures we consider are bounded, and may be placed in $\mathbb{R}^2$ or $\mathbb{R}^3$ so that any symmetry of the figure fixes the origin; that is, the origin is the "center" of the figure. Thus, any symmetry of the figure may be considered to be a homogeneous linear transformation. Hence, we recall the following result from linear algebra.

Theorem 2: Let Z ($= (0,0)$ or $(0,0,0)$) denote the origin of $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. In the case of $\mathbb{R}^2$, given two points X_1 and X_2 and their intended images Y_1 and Y_2, expressed in vector form as

$$X_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \qquad X_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} y_{21} \\ y_{22} \end{bmatrix}$$

such that Z, X_1, and X_2 form a triangle, there is a unique homogeneous linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, a linear transformation mapping Z to Z) such that $f(X_1) = Y_1$ and $f(X_2) = Y_2$. The multiplier M for f is given by

$$M = \begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}^{-1}$$

that is, f is defined by $f(X) = MX$.

2. In the case of $\mathbb{R}^3$, given three points X_1, X_2, and X_3 and their intended images Y_1, Y_2, and Y_3, expressed in vector form as

$$X_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix}, \quad Y_1 = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} y_{21} \\ y_{22} \\ y_{23} \end{bmatrix}, \quad Y_3 = \begin{bmatrix} y_{31} \\ y_{32} \\ y_{33} \end{bmatrix}$$

such that Z, X_1, X_2, and X_3 form a tetrahedron, there is a unique homogeneous linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(X_1) = Y_1$, $f(X_2) = Y_2$, and $f(X_3) = Y_3$. The multiplier M for f is given by

$$M = \begin{bmatrix} y_{11} & y_{21} & y_{31} \\ y_{12} & y_{22} & y_{32} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}^{-1}$$

that is, f is defined by $f(X) = MX$.

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Example 3: Refer to Example 1 and, in particular, to the symmetries of a square.

Exercise: Apply Theorem 2 to show that:

$$\begin{aligned} \epsilon(x,y) &= (x,y), & \rho(x,y) &= (-y,x) \\ \rho^2(x,y) &= (-x,-y), & \rho^3(x,y) &= (y,-x) \\ \sigma(x,y) &= (x,-y), & (\sigma\rho)(x,y) &= (y,x) \\ (\sigma\rho^2)(x,y) &= (-x,y), & (\sigma\rho^3)(x,y) &= (-y,-x) \end{aligned}$$

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Example 4: Refer to Example 2.

Exercise: Apply Theorem 2 to show that:

$$\begin{aligned} \rho_1(x,y,z) &= \Big((-x - \sqrt{3}\,y)/2, (\sqrt{3}\,x - y)/2, z\Big) \\ \rho_1^2(x,y,z) &= \Big((-x + \sqrt{3}\,y)/2, (-\sqrt{3}\,x - y)/2, z\Big) \\ \rho_2(x,y,z) &= (x,-y,-z) \\ \rho_3(x,y,z) &= \Big((-x + \sqrt{3}\,y)/2, (\sqrt{3}\,x + y)/2, -z\Big) \\ \rho_4(x,y,z) &= \Big((-x - \sqrt{3}\,y)/2, (-\sqrt{3}\,x + y)/2, -z\Big) \end{aligned}$$

$$\begin{aligned}
\sigma(x,y,z) &= (x,y,-z)\\
(\sigma\rho_1)(x,y,z) &= \Big((-x+\sqrt{3}\,y)/2,(\sqrt{3}\,x-y)/2,-z\Big)\\
(\sigma\rho_1^2)(x,y,z) &= \Big((-x+\sqrt{3}\,y)/2,(-\sqrt{3}\,x-y)/2,-z\Big)\\
(\sigma\rho_2)(x,y,z) &= (x,-y,z)\\
(\sigma\rho_3)(x,y,z) &= \Big((-x+\sqrt{3}\,y)/2,(\sqrt{3}\,x+y)/2,z\Big)\\
(\sigma\rho_4)(x,y,z) &= \Big((-x+\sqrt{3}\,y)/2,(-\sqrt{3}\,x+y)/2,z\Big)
\end{aligned}$$

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Additional Exercises

1. For n odd, $n \geq 3$, show that $D_n \times C_2 \cong D_{2n}$. (Refer to *Direct Products and Semidirect Products*.)

2. Prove Theorem 1.

3. Construct the Cayley digraph for

(a) D_3 (b) D_4

using generating set $\{r, s\}$.

4. Determine the symmetry group of a regular triangular pyramid that is not a tetrahedron. Hint: Place the pyramid in $\mathbb{R}^3$ so that its apex is at $A = (0,0,1)$ and the three vertices B, C, and D of the base are at the cube roots of unity in the xy-plane.

5. Construct the Cayley digraph for

(a) D_5 (b) D_6

using generating set $\{s, rs\}$.

6. Determine the symmetry group of a regular square pyramid. Hint: Place the pyramid in $\mathbb{R}^3$ so that its apex is at $A = (0,0,1)$ and the four vertices of the base are at $B = (1,0,0)$, $C = (0,1,0)$, $D = (-1,0,0)$, and $E = (0,-1,0)$ (the fourth roots of unity in the xy-plane).

7. Given a figure $\mathcal{F}$, let α and β be symmetries of $\mathcal{F}$. Show that both $\alpha\beta$ and α^{-1} are symmetries of $\mathcal{F}$.

8. Generalizing Exercises 4 and 6, show that the symmetry group of a regular n-pyramid, $n \geq 3$, is isomorphic to D_n.

9. Consider a regular n-gon, $\mathcal{P}_n$, $n \geq 3$, whose vertices are at the nth roots of unity. Erase the right half of each edge (as observed from the origin); call the resulting figure $\mathcal{F}_n$. Show that the group of symmetries of $\mathcal{F}_n$ is isomorphic to C_n.

It can be shown that, for any bounded subset $\mathcal{F}$ of $\mathbb{R}^2$, the group of symmetries of $\mathcal{F}$ is isomorphic to C_n or to D_n for some positive integer n.

10. The purpose of this exercise is to describe the group $\mathcal{G}$ of symmetries of a square prism $\mathcal{R}_4$ that is not a cube. Without loss of generality, we place the square prism in $\mathbb{R}^3$ so that its vertices are:

$$\begin{array}{llll} A=(1,0,1), & B=(0,1,1), & C=(-1,0,1), & D=(0,-1,1) \\ E=(1,0,-1), & F=(0,1,-1), & G=(-1,0,-1), & H=(0,-1,-1) \end{array}$$

(a) Any symmetry of $\mathcal{R}_4$ must map the top face $ABCD$ to itself and the bottom face $EFGH$ to itself, or must interchange these two faces. Use this observation to show that $\mathcal{R}_4$ has at most 16 symmetries.

(b) Note that the z-axis is an axis of 4-fold rotational symmetry, and that there are four axes of 2-fold rotational symmetry, all lying in the xy-plane. Use this observation to list the eight rotational symmetries of $\mathcal{R}_4$. In particular, let

$$\begin{aligned} \rho_1 &= \text{rotation}, 90^\circ, z\text{-axis} = (A\,B\,C\,D)(E\,F\,G\,H) \\ \rho_2 &= \text{rotation}, 180^\circ, x\text{-axis} = (A\,E)(B\,H)(C\,G)(D\,F) \end{aligned}$$

(c) Letting $\mathcal{H}$ denote the subgroup of rotational symmetries, show that $\mathcal{H} = \langle \rho_1, \rho_2 \rangle \cong D_4$.

(d) The reflection σ whose mirror is the xy-plane is a non-rotational symmetry of $\mathcal{R}_4$. Therefore, we see that $|\mathcal{G}| = 16$, that $\mathcal{H} \triangleleft \mathcal{G}$, and that $\mathcal{G} = \mathcal{H} \cup \sigma\mathcal{H}$. List the eight non-rotational symmetries of $\mathcal{R}_4$.

(e) Verify that $\sigma\rho_1 = \rho_1\sigma$ and that $\sigma\rho_2 = \rho_2\sigma$. It follows that

$$\mathcal{G} \cong D_4 \times C_2$$

11. The purpose of this exercise is to describe the group $\mathcal{G}$ of symmetries of the n-prism $\mathcal{R}_n$, $n \geq 5$. Without loss of generality, we place the n-prism in $\mathbb{R}^3$ so that the vertices $A_0, A_1, \ldots, A_{n-1}$ in the top face are at the nth roots of unity in the plane $z = 1$, and the vertices $B_0, B_1, \ldots, B_{n-1}$ in the bottom face are at the nth roots of unity in the plane $z = -1$, with $\overline{A_i B_i}$ a lateral edge of the n-prism for $0 \leq i \leq n-1$.

(a) Any symmetry of $\mathcal{R}_n$ must map the top face $A_0 A_1 \cdots A_{n-1}$ to itself and the bottom face $B_0 B_1 \cdots B_{n-1}$ to itself, or must interchange these two faces. Use this observation to show that $\mathcal{R}_n$ has at most $4n$ symmetries.

(b) Note that the z-axis is an axis of n-fold rotational symmetry, and that there are n axes of 2-fold rotational symmetry, all lying in the xy-plane. Use this observation to show that $\mathcal{R}_n$ has $2n$ rotational symmetries. In particular, let

$$\begin{aligned} \rho_1 &= \text{rotation}, (360/n)^\circ, z\text{-axis} = (A_0\,A_1 \cdots A_{n-1})(B_0\,B_1 \cdots B_{n-1}) \\ \rho_2 &= \text{rotation}, 180^\circ, x\text{-axis} \end{aligned}$$

(c) Letting $\mathcal{H}$ denote the subgroup of rotational symmetries, show that $\mathcal{H} = \langle \rho_1, \rho_2 \rangle \cong D_n$.

The reflection σ whose mirror is the xy-plane is a non-rotational symmetry of $\mathcal{R}_n$. Therefore, we see that $|\mathcal{G}| = 4n$, that $\mathcal{H} \lhd \mathcal{G}$, and that $\mathcal{G} = \mathcal{H} \cup \sigma\mathcal{H}$. Thus, the $2n$ non-rotational symmetries of $\mathcal{R}_n$ are the elements of $\sigma\mathcal{H}$.

(e) Verify that $\sigma\rho_1 = \rho_1\sigma$ and that $\sigma\rho_2 = \rho_2\sigma$. It follows that

$$\mathcal{G} \cong D_n \times C_2$$

Hence, if n is odd, then $\mathcal{G} \cong D_{2n}$.