Abelian Groups of Order 12

Up to isomorphism, there are two abelian groups of order 12:

A12.1: $\mathbb{Z}_{12} \cong U_{13} \cong U_{26}$

A12.2: $\mathbb{Z}_6 \times \mathbb{Z}_2 \cong U_{21} \cong U_{28} \cong U_{36} \cong U_{42}$

(See Abelian Groups for more information.)

Nonabelian Groups of Order 12

Up to isomorphism, there are three nonabelian groups of order 12:

N12.1: $D_6$ (See Dihedral Groups for more information.)

N12.2: $A_4$, the alternating group of order 12: This group has the presentation

$A_4 = \langle a, b \mid |a| = 3, |b| = 2, aba = ba^2b \rangle$

Cayley digraph using generating set $\{a, b\}$:
Order Profile:

<table>
<thead>
<tr>
<th>order</th>
<th>elements</th>
<th>number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>e</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$aba^2, a^2ba, b$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>$a, a^2, ab, a^2b, aba, a^2ba^2, ba, ba^2$</td>
<td>8</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>

Lattice of Subgroups:

Note: Nodes denoted with an inverted triangle represent nontrivial normal subgroups. Thus, $A_4$ has a unique nontrivial normal subgroup, and it is the 2-Sylow subgroup of $A_4$ (see Sylow Theorems).

Nontrivial Conjugacy Classes:

$[a] = \{a, ab, ba, a^2ba^2\}$  
$[b] = \{b, aba^2, a^2ba\}$  
$[a^2] = \{a^2, a^2b, ba^2, aba\}$

Automorphism Group: $\mathcal{A}(A_4) \cong S_4$
Inner Automorphism Group: $\mathcal{I}(A_4) \cong A_4$

Other Representations:

$A_4$ is often defined as the subgroup of $S_4$ consisting of the even permutations of \{1, 2, 3, 4\}:

$$A_4 = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 2\ 3), (1\ 2\ 4), (1\ 3\ 2), (1\ 3\ 4), (1\ 4\ 2), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\}$$

Refer to *Permutation Groups* for more information.

$A_4$ is isomorphic to the subgroup of $GL_3(\mathbb{R})$ generated by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$A_4$ is isomorphic to the semi-direct product of $\mathbb{Z}_3$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ using the homomorphism $\theta: \mathbb{Z}_3 \to \mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ defined by $\theta(1) = \phi$, with $\phi$ the automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that maps $(0, 1)$ to $(1, 0)$ and $(1, 0)$ to $(1, 1)$. To verify this, let $a = (1, (0, 0))$ and $b = (0, (1, 0))$ in $\mathbb{Z}_3 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$; show that $|a| = 3$, $|b| = 2$, and $aba = ba^2b$. (Refer to *Direct Products and Semi-direct Products* for more information.)

N12.3: The group $T$, having presentation

$$T = \langle a, b \mid |a| = 6, |b| = 4, a^3 = b^2, ba = a^5b \rangle$$

Cayley digraph using generating set \{a, b\}:
Order Profile:

<table>
<thead>
<tr>
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<th>number</th>
</tr>
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<tr>
<td>1</td>
<td>e</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$a^3 = b^2$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$a^2, a^4$</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$ab, a^2b, a^3b, a^4b, a^5b, b$</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>$a, a^5$</td>
<td>2</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>

Lattice of Subgroups:

Note: Nodes denoted with an inverted triangle represent nontrivial normal subgroups. Thus, $T$ has three nontrivial normal subgroups: $\langle a \rangle$, which has index 2; $\langle a^2 \rangle$, which is the unique 3-Sylow subgroup of $T$ (see Sylow Theorems); and $\langle a^3 \rangle$, which is the center of $T$. 
Nontrivial Conjugacy Classes:

\[ [a] = \{a, a^5\} \quad [a^2] = \{a^2, a^4\} \quad [b] = \{b, a^2b, a^4b\} \quad [ab] = \{ab, a^3b, a^5b\} \]

Automorphism Group: \( A(T) \cong D_6 \)

Inner Automorphism Group: \( I(T) \cong D_3 \)

Other Representations:

- \( T \) has the alternate presentation
  \[ T = \langle s, t \mid |s| = 4, |t| = 3, ts = st^2 \rangle \]

- \( T \) is isomorphic to the subgroup of \( S_7 \) generated by
  \[ \alpha = (1 2 3)(4 5)(6 7) \quad \text{and} \quad \beta = (1 2)(4 6 5 7) \]

Refer to *Permutation Groups* for more information.

**Exercise:** Is \( T \) isomorphic to a subgroup of \( S_{n'} \) for \( n' = 5 \) or for \( n' = 6 \)?

- \( T \) is isomorphic to the subgroup of \( GL_2(\mathbb{C}) \) generated by
  \[ S = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix} \]

  with \( w = (-1 + \sqrt{3}i)/2 \).

- \( T \) is isomorphic to the semi-direct product of \( \mathbb{Z}_4 \) with \( \mathbb{Z}_3 \) using the homomorphism \( \theta : \mathbb{Z}_4 \to A(\mathbb{Z}_3) \) defined by \( \theta(1) = \phi \), with \( \phi \) the inverse automorphism of \( \mathbb{Z}_3 \). To verify this, let \( s = (1, 0) \) and \( t = (0, 1) \) in \( \mathbb{Z}_4 \times \mathbb{Z}_3 \); show that \( |s| = 4, |t| = 3, \) and \( ts = st^2 \). (Refer to *Direct Products and Semi-direct Products* for more information.)

In what follows, we prove that \( D_6, A_4, \) and \( T \) are the only nonabelian groups of order 12, up to isomorphism. To that end, let \( G \) be a nonabelian group of order 12 with identity \( e \). The proof uses the *Sylow Theorems* and semi-direct products (refer to *Direct Products and Semi-direct Products*).

Let \( s_2 \) and \( s_3 \) denote the number of distinct 2-Sylow subgroups and 3-Sylow subgroups of \( G \), respectively. Then, by Sylow’s counting theorem,

\[ s_2 \in \{1, 3\} \quad \text{and} \quad s_3 \in \{1, 4\} \]

Since \( G \) is nonabelian, it is not the case that \( s_2 = 1 \) and \( s_3 = 1 \). We claim that, if \( s_3 = 4 \), then \( s_2 = 1 \).

To verify this claim, suppose \( s_3 = 4 \). Then \( G \) has four distinct 3-Sylow subgroups, and this implies that \( G \) contains exactly eight elements of order 3. This leaves only \( e \) and three other elements, which must constitute the unique 2-Sylow subgroup of \( G \) (which has order 4). Therefore, \( s_2 = 1 \).
Thus, there are two cases to consider: (1) $s_3 = 1$ and $s_2 = 3$; and (2) $s_3 = 4$ and $s_2 = 1$.

Case 1: $s_3 = 1$ and $s_2 = 3$. Let $N$ be the unique 3-Sylow subgroup of $G$. Thus, $N \triangleleft G$. Also, since $|N| = 3$, $N \cong \mathbb{Z}_3$. We then consider two subcases depending on the structure of the 2-Sylow subgroups.

Subcase 1A: $G$ contains an element of order 4, and hence has a 2-Sylow subgroup isomorphic to $\mathbb{Z}_4$. In this case, $G$ is isomorphic to a semi-direct product of $\mathbb{Z}_4$ with $\mathbb{Z}_3$ using $\theta$, where $\theta$ is a nontrivial homomorphism from $\mathbb{Z}_4$ to the automorphism group $\mathcal{A}(\mathbb{Z}_3)$ of $\mathbb{Z}_3$. There are only two automorphisms of $\mathbb{Z}_3$: the identity map $\epsilon$ and the inverse map $\phi$ defined by $\phi(x) = -x$. Therefore,

$$
\theta_0 = \epsilon, \quad \theta_1 = \phi, \quad \theta_2 = \epsilon, \quad \theta_3 = \phi
$$

In this semi-direct product, let $s = (1, 0)$ and $t = (0, 1)$.

**Exercise:** Verify that $|s| = 4$, $|t| = 3$, and $ts = st^2$. It follows that $G$ is isomorphic to the group $T$.

Subcase 1B: $G$ contains no elements of order 4; hence each 2-Sylow subgroup of $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, $G$ is isomorphic to a semi-direct product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}_3$ using $\theta$, where $\theta$ is a nontrivial homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the automorphism group $\mathcal{A}(\mathbb{Z}_3)$ of $\mathbb{Z}_3$. Again, there are only two automorphisms of $\mathbb{Z}_3$: the identity map $\epsilon$ and the inverse map $\phi$ defined by $\phi(x) = -x$, and any homomorphism $\theta$ must map exactly two of $(1,0), (0,1), (1,1)$ to $\phi$. By symmetry, we may assume that

$$
\theta_{(0,0)} = \epsilon, \quad \theta_{(1,0)} = \epsilon, \quad \theta_{(0,1)} = \phi, \quad \theta_{(1,1)} = \phi
$$

In this semi-direct product, let $r = ((1,0), 1)$ and $s = ((0,1), 0)$.

**Exercise:** Verify that $|r| = 6$, $|s| = 2$, $s \neq r^3$, and $sr = r^5s$. It follows that $G$ is isomorphic to the group $D_6$.

Case 2: $s_3 = 4$ and $s_2 = 1$. Let $N$ be the unique 2-Sylow subgroup of $G$ and let $H$ be a 3-Sylow subgroup of $G$. Then $N \triangleleft G$, $HN = G$, and $H \cap N = \{e\}$. Hence, $G$ is isomorphic to a semi-direct product of $\mathbb{Z}_3$ with $N$.

Is it possible that $N$ is isomorphic to $\mathbb{Z}_4$? Note that the only homomorphism from $\mathbb{Z}_3$ to $\mathbb{Z}_4$ is the trivial homomorphism (since 1 is of order 3 and $\mathcal{A}(\mathbb{Z}_4)$ has order 2). Any semi-direct product using the trivial homomorphism is a direct product, and any direct product of abelian groups is abelian. So, the answer to our question is no.

Therefore, $N$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $G$ is isomorphic to a semi-direct product of $\mathbb{Z}_3$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ using $\theta$, where $\theta$ is a nontrivial homomorphism from $\mathbb{Z}_3$ to the automorphism group $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$. 

Now then, $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$, since any permutation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that fixes $(0,0)$ is an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Among these, there are two of order 3: $\phi$, mapping $(0,1)$ to $(1,0)$, $(1,0)$ to $(1,1)$, and $(1,1)$ to $(0,1)$, and $\phi^{-1}$. We leave it as an exercise to show that the homomorphism $\theta$ that maps 1 to $\phi$, and the homomorphism $\theta'$ that maps 1 to $\phi^{-1}$, yield isomorphic semi-direct products. Thus, we take $\theta$ to be the homomorphism defined by

$$\theta_0 = \epsilon, \quad \theta_1 = \phi, \quad \theta_2 = \phi^{-1}$$

As shown above, the semi-direct product in this case is isomorphic to the group $A_4$. 

