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Smokin' Joe's Catalog of Groups: Groups of Orders 28 – 31

Abelian Groups of Order 28

Up to isomorphism, there are two abelian groups of order 28:

A28.1: $\mathbb{Z}_{28} \cong U_{29} \cong U_{58}$

A28.2: $\mathbb{Z}_{14} \times \mathbb{Z}_2$

(See *Abelian Groups* for more information.)

Nonabelian Groups of Order 28

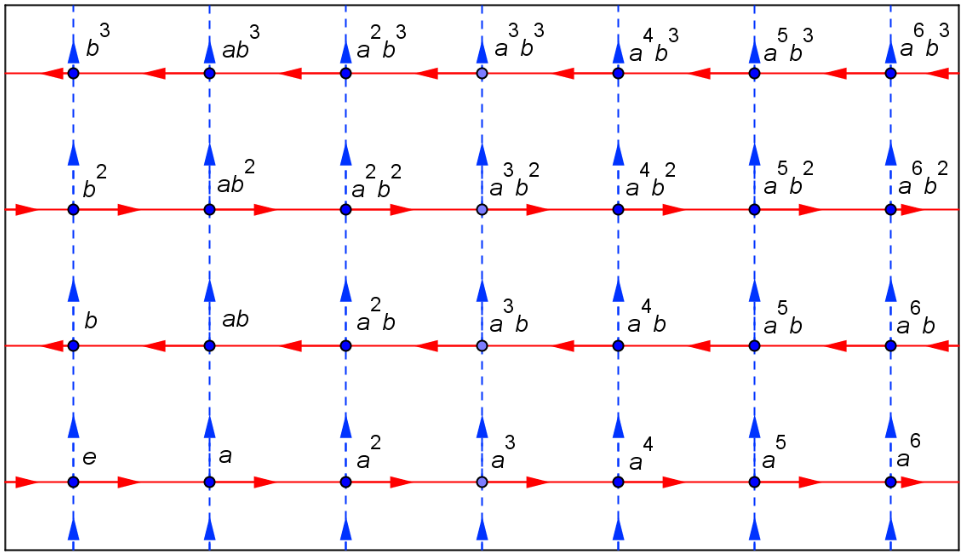
Up to isomorphism, there are two nonabelian groups of order 28:

N28.1: D_{14} (See *Dihedral Groups* for more information.)

N28.2: The group G_2 with presentation

$$\langle a, b \mid |a| = 7, |b| = 4, ba = a^6 b \rangle$$

Cayley digraph using generating set $\{a, b\}$, shown on a torus:



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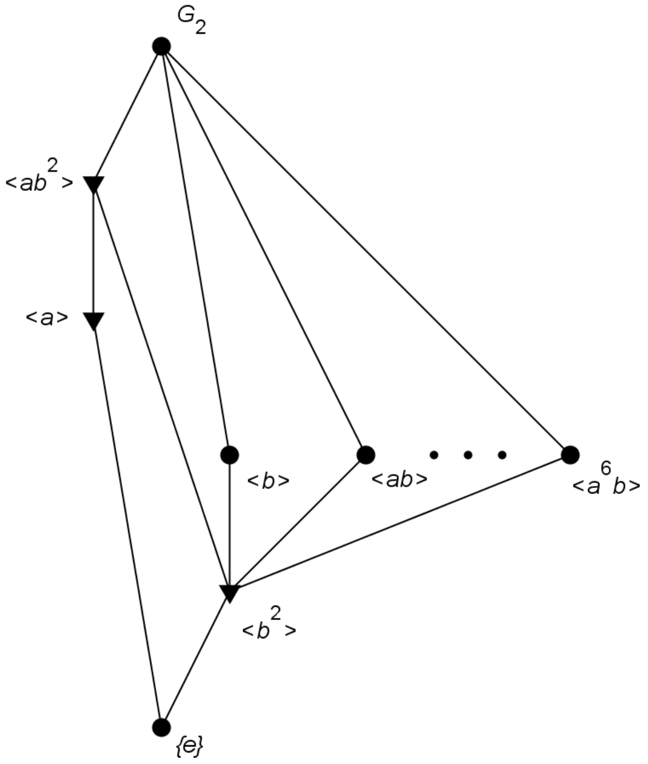
Order Profile:

order	elements	number
1	e	1
2	b^2	1
4	rest	14
7	$a, a^2, a^3, a^4, a^5, a^6$	6
14	$ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2$	6
total		28

Lattice of Subgroups:



G_2 has three nontrivial normal subgroups: $\langle ab^2\rangle$, which is the unique subgroup of G_2 of order 14; $\langle a\rangle$, which is the unique 7-Sylow subgroup, and is also the commutator subgroup of G_2; and $\langle b^2\rangle$, which is the center $\mathcal{C}$ of G_2.

Nontrivial Conjugacy Classes:

$$\begin{array}{rlll}
[b] &= \{b, ab, a^2b, a^3b, a^4b, a^5b, a^6b\} & & \\
[b^3] &= \{b^3, ab^3, a^2b^3, a^3b^3, a^4b^3, a^5b^3, a^6b^3\} & & \\
[a] &= \{a, a^6\} & [a^2] = \{a^2, a^5\} & [a^3] = \{a^3, a^4\} \\
[ab^2] &= \{ab^2, a^6b^2\} & [a^2b^2] = \{a^2b^2, a^5b^2\} & [a^3b^2] = \{a^3b^2, a^4b^2\}
\end{array}$$

Automorphism Group: $|\mathcal{A}(G_2)| = 84$. The following recipe unique determines an automorphism ϕ of G_2:

Step 1: Choose one of the 6 elements of order 7 for $\phi(a)$.

Step 2: Choose one of the 14 elements of order 4 for $\phi(b)$.

Inner Automorphism Group: $\mathcal{I}(G_2) \cong G_2/\mathcal{C} \cong D_7$.

Other Representations:

G_2 is isomorphic to the subgroup of $GL_2(\mathbb{C})$ (actually, $SL_2(\mathbb{C})$) generated by

$$A = \begin{bmatrix} w & 1 \\ -1 & w^6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with $w = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$.

G_2 is isomorphic to the semi-direct product of $\mathbb{Z}_4$ with $\mathbb{Z}_7$ using θ, where $\theta : \mathbb{Z}_4 \to \mathcal{A}(\mathbb{Z}_7)$ is the homomorphism that maps 1 to the automorphism ϕ that maps each element of $\mathbb{Z}_7$ to its inverse. (Refer to *Direct Products and Semi-direct Products*; in particular, to Exercise 42.)

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In what follows, we outline a proof that D_{14} and G_2 are the only nonabelian groups of order 28, up to isomorphism. To that end, let G be a nonabelian group of order 28 with identity e. The proof uses the *Sylow Theorems* and semi-direct products.

Let s_7 be the number of 7-Sylow subgroups of G. Then $s_7 \equiv 1 \pmod 7$ and $s_7 \mid 28$, so it follows that $s_7 = 1$. Thus, G has a unique 7-Sylow subgroup N, and so N is a normal subgroup of G.

We then consider two cases depending on the structure of the 2-Sylow subgroups.

Case 1: Some 2-Sylow subgroup is isomorphic to $\mathbb{Z}_4$.

Case 2: Every 2-Sylow subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise: Let s_2 denote the number of 2-Sylow subgroups of G. Show that $s_2 = 7$.

Let H be a 2-Sylow subgroup of G. Then $G = HN$ and $H \cap N = \{e\}$, and it follows that G is isomorphic to a semi-direct product of a group of order 4 with $\mathbb{Z}_7$.

Note that this semi-direct product uses a nontrivial homomorphism θ from H to the automorphism group $\mathcal{A}(\mathbb{Z}_7)$ of $\mathbb{Z}_7$, and that $\mathcal{A}(\mathbb{Z}_7)$ has no elements of order 4 and a unique element of order 2, namely, the automorphism ϕ of $\mathbb{Z}_7$ defined by $\phi(x) = -x$. We now consider the two cases:

Case 1: H is cyclic. Then G is isomorphic to the semi-direct product of $\mathbb{Z}_4$ with $\mathbb{Z}_7$ using θ, where θ is the nontrivial homomorphism from $\mathbb{Z}_4$ to $\mathcal{A}(\mathbb{Z}_7)$ defined by $\theta(1) \ = \phi$. Therefore, $G \cong G_2$.

Case 2: $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then G is isomorphic to the semi-direct product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}_7$ using θ, where θ is a nontrivial homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathcal{A}(\mathbb{Z}_7)$. Such a homomorphism must map exactly two of $(0, 1)$, $(1, 0)$, and $(1, 1)$ to ϕ. Without loss of generality, we assume that

$$\theta(0, 1) = \phi = \theta(1, 1)$$

Exercise: Show, in this case, that G is isomorphic to D_{14}.

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Groups of Order 29

Since 29 is prime, there is a unique group of order 29, up to isomorphism, $\mathbb{Z}_{29}$. Refer to *Cyclic Groups* for more information.

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Abelian Groups of Order 30

Up to isomorphism, there is a unique abelian group of order 30:

A30.1: $\mathbb{Z}_{30} \cong U_{31} \cong U_{62}$

(See *Abelian Groups* for more information.)

Nonabelian Groups of Order 30

Up to isomorphism, there are three nonabelian groups of order 30:

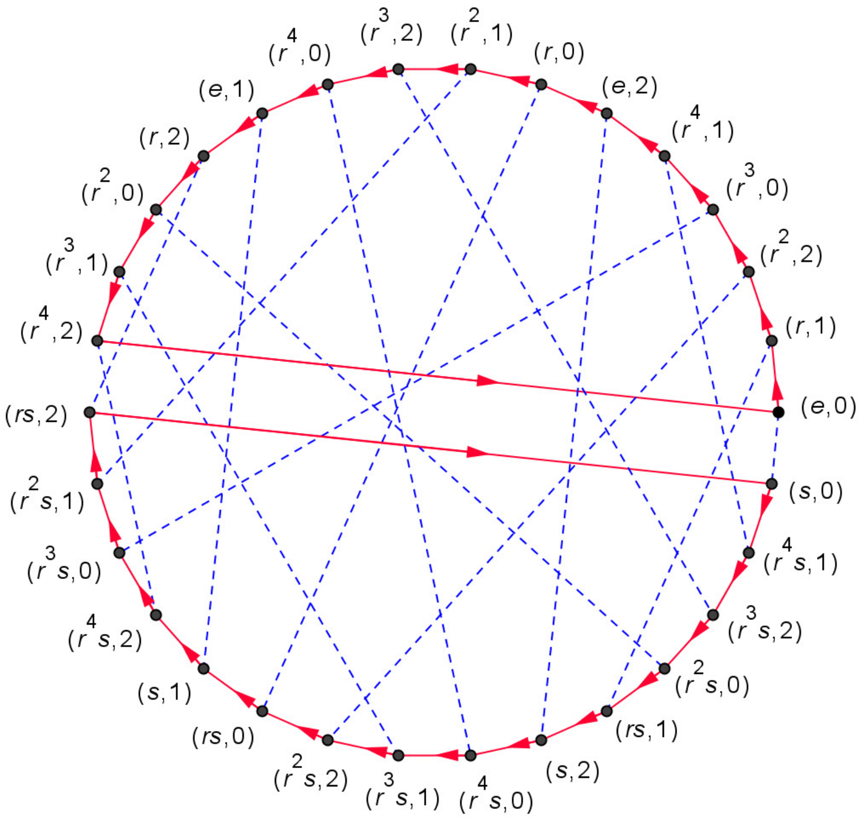
N30.1: D_{15} (See *Dihedral Groups* for more information.)

N30.2: $G_2 = D_5 \times \mathbb{Z}_3$

In discussing this group, we assume that D_5 has the presentation

$$\langle r, s \mid |r| = 5, |s| = 2, sr = r^4 s \rangle$$

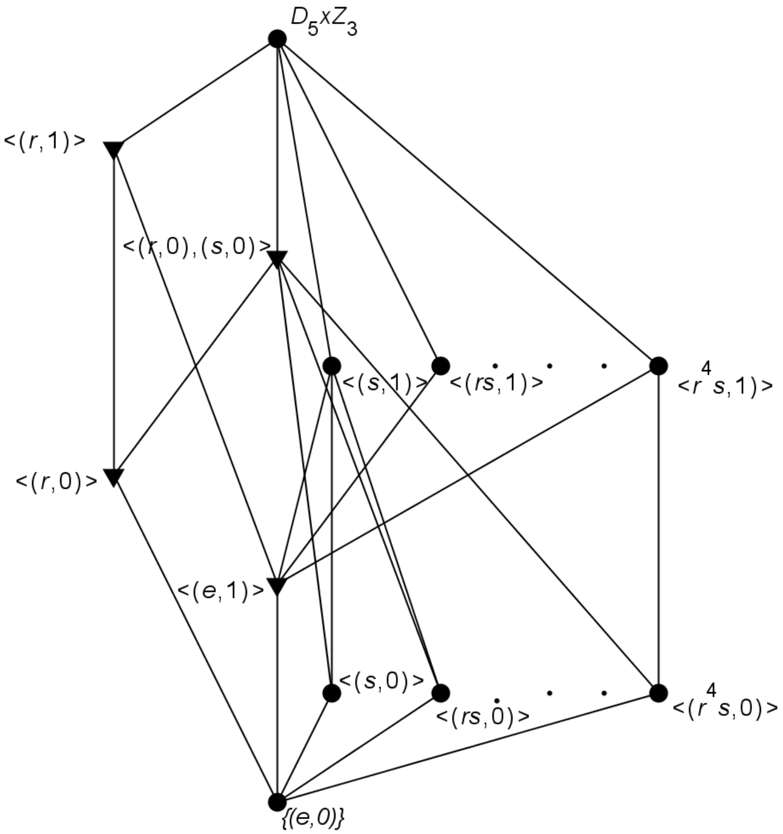
Cayley digraph using generating set $\{(r,1),(s,0)\}$:



Order Profile:

order	elements	number
1	$(e,0)$	1
2	$(s,0),(rs,0),(r^2s,0),(r^3s,0),(r^4s,0)$	5
3	$(e,1),(e,2)$	2
5	$(r,0),(r^2,0),(r^3,0),\ (r^4,0)$	4
6	$(s,1),(rs,1),(r^2s,1),(r^3s,1),(r^4s,1),$ $(s,2),(rs,2),(r^2s,2),(r^3s,2),(r^4s,2)$	10
15	$(r,1),(r^2,1),(r^3,1),\ (r^4,1),$ $(r,2),(r^2,2),(r^3,2),\ (r^4,2)$	8
total		30

Lattice of Subgroups:



$D_5 \times \mathbb{Z}_3$ has four nontrivial normal subgroups: $\langle (r,1) \rangle$, which has index 2; $\langle (r,0),(s,0) \rangle \cong D_5$; $\langle (r,0) \rangle$, which is the commutator subgroup, and also the unique 5-Sylow subgroup; and $\langle (e,1) \rangle$, which the center $\mathcal{C}$ of $D_5 \times \mathbb{Z}_3$.

Nontrivial Conjugacy Classes: If $[x]$ is a nontrivial conjugacy class of D_5, then $\{x\} \times \{0,1,2\}$ is a nontrivial conjugacy class of $D_5 \times \mathbb{Z}_3$.

Automorphism Group: $|\mathcal{A}(D_5 \times \mathbb{Z}_3)| = 40$. The following recipe uniquely determines an automorphism ϕ of $D_5 \times \mathbb{Z}_3$:

1. Choose $(r,0)$, $(r^2,0)$, $(r^3,0)$, or $(r^4,0)$ for $\phi(r,0)$.
2. Choose $(s,0)$, $(rs,0)$, $(r^2s,0)$, $(r^3s,0)$, or $(r^4s,0)$ for $\phi(s,0)$.
3. Choose either $(0,1)$ or $(0,2)$ for $\phi(0,1)$.

Inner Automorphism Group: $\mathcal{I}(D_5 \times \mathbb{Z}_3) \cong (D_5 \times \mathbb{Z}_3)/\mathcal{C} \cong D_5$.

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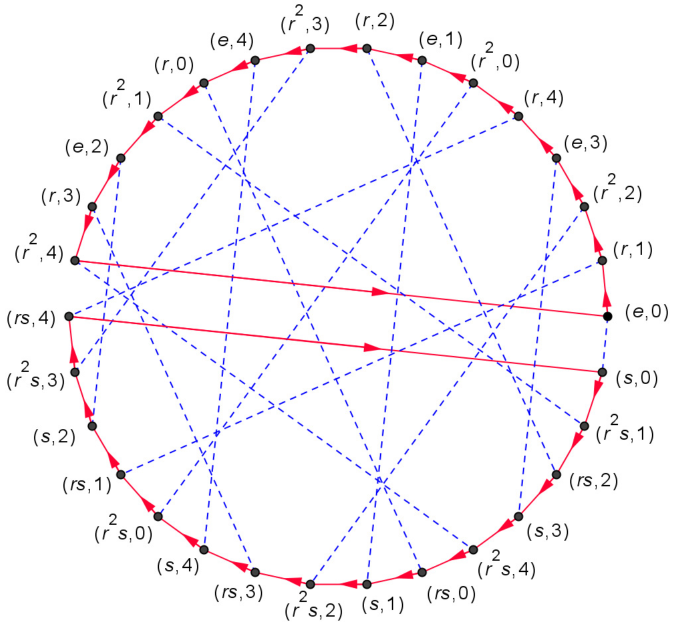
N30.3: $G_3 = D_3 \times \mathbb{Z}_5$

In discussing this group, we assume that D_3 has the presentation

$$\langle r, s \mid |r| = 3, |s| = 2, sr = r^2 s \rangle$$

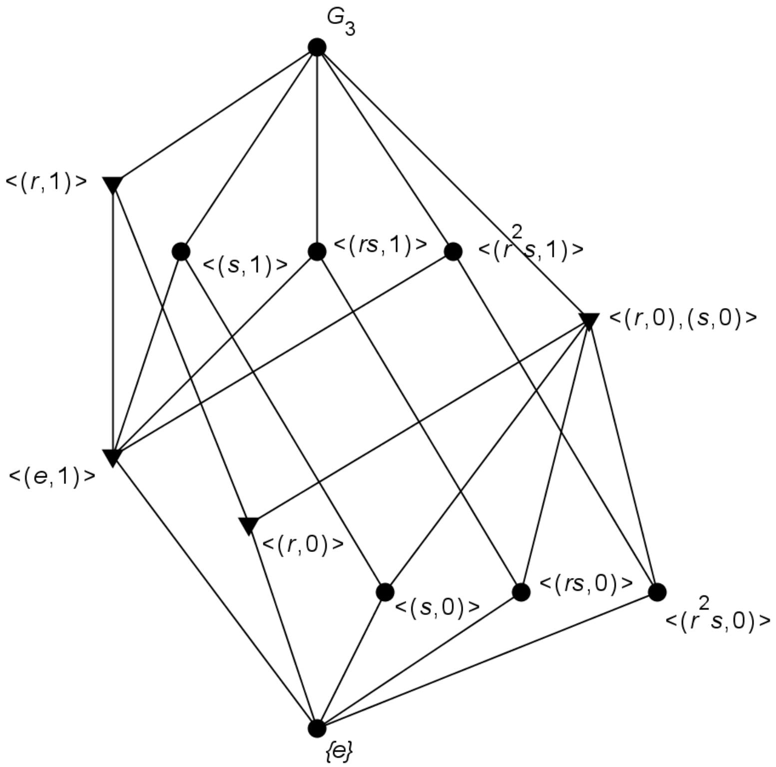
Cayley digraph using generating set $\{(r,1), (s,0)\}$:



Order Profile:

order	elements	number
1	$(e,0)$	1
2	$(s,0), (rs,0), (r^2s,0)$	3
3	$(r,0), (r^2,0)$	2
5	$(e,1), (e,2), (e,3), (e,4)$	4
10	$(s,1), (rs,1), (r^2s,1), (s,2), (rs,2), (r^2s,2),$ $(s,3), (rs,3), (r^2s,3), (s,4), (rs,4), (r^2s,4)$	12
15	$(r,1), (r^2,1), (r,2), (r^2,2),$ $(r,3), (r^2,3), (r,4), (r^2,4)$	8
total		30

Lattice of Subgroups:



G_3 has four nontrivial normal subgroups: $\langle(r,1)\rangle$, which has index 2; $\langle(r,0),(s,0)\rangle$, which is isomorphic to D_3; $\langle(e,1)\rangle$, which is the center $\mathcal{C}$ of G_3, and is also the unique 5-Sylow subgroup; and $\langle(r,0)\rangle$, which is the commutator subgroup, and is also the unique 3-Sylow subgroup.

Nontrivial Conjugacy Classes: If $[x]$ is a nontrivial conjugacy class of D_3, then $\{x\} \times \{0,1,2,3,4\}$ is a nontrivial conjugacy class of $D_3 \times \mathbb{Z}_5$.

Automorphism Group: $|\mathcal{A}(G_3)| = 24$. The following recipe uniquely determines an automorphism ϕ of G_3:

1. Choose either $(r,0)$ or $(r^2,0)$ for $\phi(r,0)$.
2. Choose either $(s,0)$, $(rs,0)$, or $(r^2s,0)$ for $\phi(s,0)$.
3. Choose either $(e,1)$, $(e,2)$, $(e,3)$, or $(e,4)$ for $\phi(e,1)$.

In fact, $\mathcal{A}(D_3 \times \mathbb{Z}_5) \cong D_3 \times \mathbb{Z}_4$.

Inner Automorphism Group: $\mathcal{I}(G_3) \cong G_3/\mathcal{C} \cong D_3$.

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In what follows, we outline a proof that D_{15}, G_2, and G_3 are the only nonabelian groups of order 30, up to isomorphism. To that end, let G be a nonabelian group of order 30 with identity e. The proof uses the *Sylow Theorems* and semi-direct products.

Let s_p be the number of p-Sylow subgroups of G, for $p = 2, 3, 5$. Then $s_5 \in \{1, 6\}$, $s_3 \in \{1, 10\}$, and $s_2 \in \{1, 3, 5, 15\}$.

Exercise: Argue that $s_5 = 1$ or $s_3 = 1$.

We then consider three cases.

Case 1: $s_5 = 1$ and $s_3 = 1$.

Exercise: Show that G contains an element of order 15. It follows that G is isomorphic to a semi-direct product of $\mathbb{Z}_2$ with $\mathbb{Z}_{15}$ using θ, where θ is a nontrivial homomorphism from $\mathbb{Z}_2$ to $\mathcal{A}(\mathbb{Z}_{15})$.

(a) If $\theta(1)$ is the automorphism ϕ' of $\mathbb{Z}_{15}$ that maps 1 to 4 (or, equivalently, to 11), show that $G \cong G_3$.

(b) If $\theta(1)$ is the automorphism ϕ of $\mathbb{Z}_{15}$ that maps 1 to 14, show that $G \cong D_{15}$.

Case 2: $s_5 = 1$ and $s_3 = 10$. Then G has a unique 5-Sylow subgroup — call it N; so N is a normal subgroup of G and N is isomorphic to $\mathbb{Z}_5$. Let b and c be elements of G of orders 2 and 3, respectively, and let $H = \langle b, c \rangle$. Then $|H| = 6$, $G = HN$ and $H \cap N = \{e\}$, so that G is isomorphic to a semi-direct product of H with N.

Exercise: If $H \cong \mathbb{Z}_6$, then G is isomorphic to a semi-direct product of $\mathbb{Z}_6$ with $\mathbb{Z}_5$ using θ, where θ is a nontrivial homomorphism from $\mathbb{Z}_6$ to $\mathcal{A}(\mathbb{Z}_5)$. Show that $\theta(1)$ is the automorphism ϕ of $\mathbb{Z}_5$ defined by $\phi(x) = -x$, and hence that G is isomorphic to G_2.

Exercise: If $H \cong D_3$, then G is isomorphic to a semi-direct product of D_3 with $\mathbb{Z}_5$ using θ, where θ is a nontrivial homomorphism from D_3 to $\mathcal{A}(\mathbb{Z}_5)$. Show that $\theta(r)$ is the identity automorphism of $\mathbb{Z}_5$ and $\theta(s)$ is the automorphism ϕ of $\mathbb{Z}_5$ defined by $\phi(x) = -x$, and hence that G is isomorphic to D_{15}.

Case 3: $s_3 = 1$ and $s_5 = 6$. Then G has a unique 3-Sylow subgroup — call it M; so M is a normal subgroup of G and M is isomorphic to $\mathbb{Z}_3$, say $M = \langle b \rangle$. Also, let a and c be elements of orders 5 and 2 in G, respectively.

Exercise: Show that G contains 24 elements of order five, 2 elements of order three, and either 3 elements of order two, or 1 element of order two and 2 elements of order six.

Let $H = \langle a \rangle$. Since G has no elements of order 15, it must be that $G = HN$, and $H \cap N = \{e\}$. But then G is isomorphic to a semi-direct product of $\mathbb{Z}_5$ with $\mathbb{Z}_3$, which is a contradiction. So this case can't occur.

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Groups of Order 31

Since 31 is prime, there is a unique group of order 31, up to isomorphism, $\mathbb{Z}_{31}$. Refer to *Cyclic Groups* for more information.

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