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Smokin' Joe's Catalog of Groups: Groups of Order 8

Abelian Groups of Order 8

Up to isomorphism, there are three abelian groups of order 8:

A8.1: $\mathbb{Z}_8 \cong U_{16}$

A8.2: $\mathbb{Z}_4 \times \mathbb{Z}_2 \cong U_{15} \cong U_{20} \cong U_{30}$

A8.3: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong U_{24}$

(See *Abelian Groups* for more information.)

Nonabelian Groups of Order 8

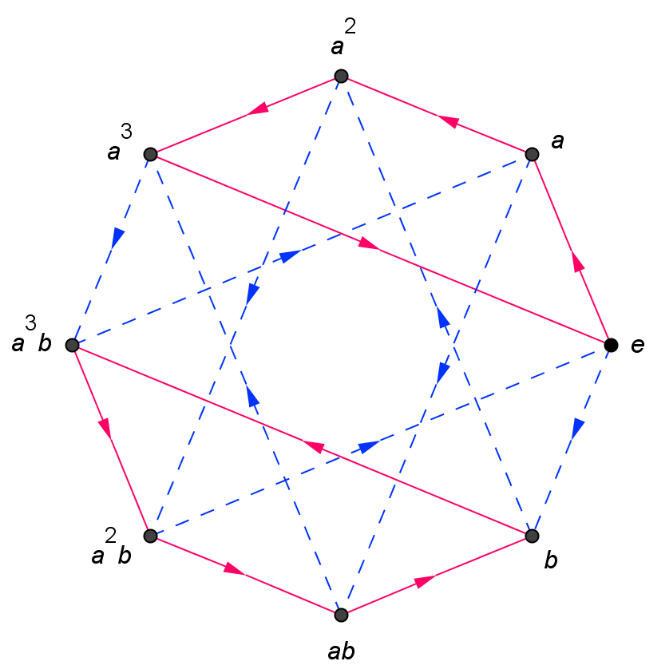
Up to isomorphism, there are two nonabelian groups of order 8:

N8.1: D_4 (See *Dihedral Groups* for more information.)

N8.2: Q, the *quaternion group*

$$Q = \langle a, b \mid |a| = 4, |b| = 4, b \notin \langle a \rangle, a^2 = b^2, ba = a^3 b \rangle$$

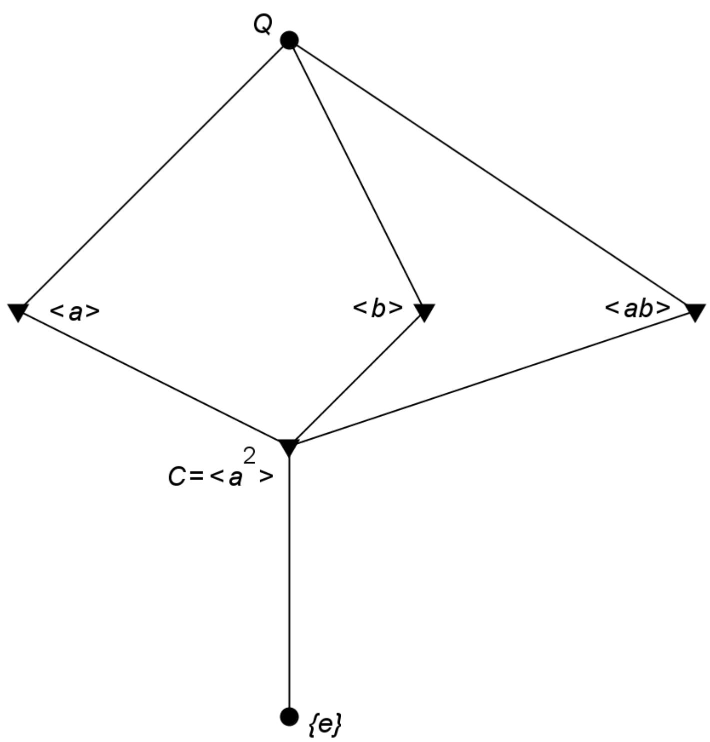
Cayley digraph using generating set $\{a, b\}$:



Order Profile:

order	elements	number
1	e	1
2	$a^2 = b^2$	1
4	rest	6
total		8

Lattice of Subgroups:



Note: Nodes denoted with an inverted triangle represent nontrivial normal subgroups. Thus, every subgroup of Q is normal.

Nontrivial Conjugacy Classes:

$$[a] = \{a, a^3\} \qquad [b] = \{b, b^3\} \qquad [ab] = \{ab, a^3b\}$$

Automorphism Group: $\mathcal{A}(Q) \cong S_4$

Inner Automorphism Group: $\mathcal{I}$(Q) $\cong$ Q/$\mathcal{C}_Q \cong Q/\langle a^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Other Representations:

Q is isomorphic to the subgroup of $GL_2(\mathbb{C})$ generated by

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

By Cayley's Theorem (see *Permutation Groups*), Q is isomorphic to a subgroup of S_8. In fact, using the proof of Cayley's Theorem, we obtain that Q is isomorphic to the subgroup of S_8 generated by

$$\alpha = (1\,2\,3\,4)(5\,6\,7\,8) \qquad \text{and} \qquad \beta = (1\,5\,3\,7)(2\,8\,4\,6)$$

Exercise: Is Q isomorphic to a subgroup of $S_{n'}$ for some n', $4 \leq n' < 8$?

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In what follows, we prove that D_4 and Q are the only nonabelian groups of order 8, up to isomorphism.

Let G be a nonabelian group of order 8, and let a be an element of G of maximum order. Clearly, $|a| \neq 8$ since, if $|a| = 8$, then G is cyclic, and hence abelian.

Is it possible that $|a| = 2$?

Exercise: Let G be a finite group with identity e. Show that, if every nonidentity element of G has order 2, then G is abelian.

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Hence, $|a| = 4$. Let $H = \langle a \rangle$, and choose $b \in G - H$ to have the minimum order among the elements of $G - H$. So

$$G = H \cup Hb = \{e, a, a^2, a^3\} \cup \{b, ab, a^2b, a^3b\}$$

We now consider two subcases:

Subcase (a): $|b| = 4$. Then G has a unique element of order 2, and so $a^2 = b^2$.

Exercise: Show that $ba = a^3b$. Thus, a presentation for G is

$$G = \langle a, b \mid |a| = 4, |b| = 4, b \notin \langle a \rangle, a^2 = b^2, ba = a^3b \rangle$$

It follows that G is isomorphic to Q.

Subcase (b): $|b| = 2$.

Exercise: Show that $ba = a^3b$. Thus, a presentation for G is

$$G = \langle a, b \mid |a| = 4, |b| = 2, b \notin \langle a \rangle, ba = a^3b \rangle$$

Thus, $G \cong D_4$.