



$$p := x \rightarrow \prod_{i=1}^n (x - r_i) \quad (10)$$

Here, we've defined  $p$  to be a function, rather than an expression. (To get the arrow, type a hyphen followed by a "greater than" symbol).

Let's expand  $p(x)$  and see what we get.

```
> collect(expand(p(x)), x);
```

$$x^4 + (-r_2 - r_1 - r_4 - r_3) x^3 + (r_1 r_2 + r_2 r_4 + r_2 r_3 + r_1 r_4 + r_1 r_3 + r_3 r_4) x^2 + (-r_1 r_2 r_4 - r_1 r_2 r_3 - r_2 r_3 r_4 - r_1 r_3 r_4) x + r_1 r_2 r_3 r_4 \quad (11)$$

If you want to see what happens for  $n = 5$ , just go back to (9), change  $n$  to 5, and then re-execute (10) and (11). At this point you should have enough information to state Vieta's theorem.

Here's another example using `expand` and `factor`. This one relates to the binomial theorem and uses the `sum` function.

```
> n := 5;
```

$$n := 5 \quad (12)$$

```
> p := (x, y) → sum( (binomial(n, k) * x^(n-k) * y^k), k=0..n );
```

$$p := (x, y) \rightarrow \sum_{k=0}^n \text{binomial}(n, k) x^{n-k} y^k \quad (13)$$

```
> factor(p(x, y));
```

$$(x + y)^5 \quad (14)$$

```
> expand(%);
```

$$x^5 + 5 x^4 y + 10 x^3 y^2 + 10 x^2 y^3 + 5 x y^4 + y^5 \quad (15)$$

The `factor` function has an optional second argument, indicating the field over which the factorization is to be done. The next several examples illustrate this.

```
> p := x → (2*x - 1) * (x^2 - 2) * (x^2 + 1);
```

$$p := x \rightarrow (2x - 1) (x^2 - 2) (x^2 + 1) \quad (16)$$

```
> expr := collect(expand(p(x)), x);
```

$$\text{expr} := 2x^5 - 2x^3 - 4x - x^4 + x^2 + 2 \quad (17)$$

```
> factor(expr);
```

$$(2x - 1) (x^2 - 2) (x^2 + 1) \quad (18)$$

```
> factor(expr, real);
```

$$2. (x + 1.414213562) (x - 0.5000000000) (x - 1.414213562) (x^2 + 1.) \quad (19)$$

```
> factor(expr, sqrt(2));
```

$$-(x^2 + 1) (-x + \sqrt{2}) (x + \sqrt{2}) (2x - 1) \quad (20)$$

```
> factor(expr, complex);
```

$$2. (x + 1.414213562) (x + 1. I) (x - 1. I) (x - 0.5000000000) (x - 1.414213562) \quad (21)$$

```
> factor(expr, I);
```

$$-(x^2 - 2) (x + I) (-x + I) (2x - 1) \quad (22)$$

```
> factor(expr, [sqrt(2), I]);
```

$$(21) \quad (23)$$

$$(-x + \sqrt{2})(x + \sqrt{2})(x + 1)(-x + 1)(2x - 1) \quad (23)$$

To solve equations, we employ the **solve** function. The simple version takes the form  $\text{solve}(\text{equation}, \text{variable})$  while the advanced version has the form  $\text{solve}(\text{list-of-equations}, \text{list-of-variables})$

For our first example, we find the points of intersection of  $y = 2x^3 + 2$  and  $y = x^2 + 4x$ .

$$\begin{aligned} > f := x \rightarrow x^2 + 4 \cdot x; & \quad f := x \rightarrow x^2 + 4x \end{aligned} \quad (24)$$

$$\begin{aligned} > \text{solve}(2 \cdot x^3 + 2 = f(x), x); & \quad \frac{1}{2}, \sqrt{2}, -\sqrt{2} \end{aligned} \quad (25)$$

$$\begin{aligned} > f\left(\frac{1}{2}\right); & \quad \frac{9}{4} \end{aligned} \quad (26)$$

$$\begin{aligned} > f(\text{sqrt}(2)); & \quad 2 + 4\sqrt{2} \end{aligned} \quad (27)$$

$$\begin{aligned} > f(-\text{sqrt}(2)); & \quad 2 - 4\sqrt{2} \end{aligned} \quad (28)$$

We might try solving the problem this way:

$$\begin{aligned} > \text{solve}([y = 2 \cdot x^3 + 2, y = x^2 + 4 \cdot x], [x, y]); & \quad (29) \\ \left[ \left[ x = \frac{1}{2}, y = \frac{9}{4} \right], [x = \text{RootOf}(\_Z^2 - 2, \text{label} = \_L5), y = 4 \text{RootOf}(\_Z^2 - 2, \text{label} = \_L5) \right. \right. \\ \left. \left. + 2] \right] \end{aligned}$$

Well, Maple is able to find the point of intersection  $(1/2, 9/4)$ , but then gets lazy on us.

Let's see if we have better luck finding where the line  $y = x - 5$  intersects the ellipse  $x^2 + 4y^2 = 25$ .

$$\begin{aligned} > \text{solve}([x^2 + 4 \cdot y^2 = 25, y = x - 5], [x, y]); & \quad (30) \\ & \quad [[x = 5, y = 0], [x = 3, y = -2]] \end{aligned}$$

Next, let's find the parabola that contains the points  $(-1, 10)$ ,  $(1, 0)$ , and  $(2, 4)$ . If we let the equation of the parabola be  $y = ax^2 + bx + c$ , then we obtain a system of three linear equations in the variables  $a$ ,  $b$ , and  $c$ :

$$\begin{aligned} > \text{solve}([a - b + c = 10, a + b + c = 0, 4 \cdot a + 2 \cdot b + c = 4], [a, b, c]); & \quad (31) \\ & \quad [[a = 3, b = -5, c = 2]] \end{aligned}$$

So the equation of the parabola is  $y = 3x^2 - 5x + 2$ .

Lastly, let's determine which positive irrational numbers have periodic continued fractions of the form  $a + 1/(b + 1/(c + 1/(b + 1/(c + \dots)))$

where  $a$  is a nonnegative integer and  $b$  and  $c$  are distinct positive integers.

$$\begin{aligned} > \text{solve}\left(\frac{1}{\frac{1}{x - a} - b} - c = x - a, x\right); \end{aligned}$$

$$\frac{1}{2} \frac{2ba - cb + \sqrt{c^2 b^2 + 4cb}}{b}, -\frac{1}{2} \frac{-2ba + cb + \sqrt{c^2 b^2 + 4cb}}{b} \quad (32)$$

> subs([a = 1, b = 2, c = 4], %[1]);

$$-1 + \frac{1}{4} \sqrt{96} \quad (33)$$

> simplify(%);

$$-1 + \sqrt{6} \quad (34)$$

We've only scratched the surface of what the solve command can do. For more details, use Maple "Help," and search on "solve."

The solve command is useful when we're interested in exact solutions. Often in mathematics, we only need an approximate solution to an equation. In this case, the function **fsolve** should be used.

First, we apply fsolve to approximate the unique real root of  $f(x) = x^5 + x^3 - 1$ :

> f := x → x<sup>5</sup> + x<sup>3</sup> - 1;

$$f := x \rightarrow x^5 + x^3 - 1 \quad (35)$$

> fsolve(f(x) = 0);

$$0.8376197748 \quad (36)$$

> f(%);

$$-1.10^{-10} \quad (37)$$

Next, we use fsolve to approximate the unique positive real solution to  $1 - \cos(t) = (1.5)(t - \sin(t))$ :

> fsolve(1 - cos(t) = (1.5) · (t - sin(t)));

$$0. \quad (38)$$

Well, 0 is a solution, but not the one we wanted. Fortunately, as an option to fsolve, we can specify an interval in which to search:

> fsolve(1 - cos(t) = (1.5) · (t - sin(t)), t = 1 .. 2);

$$1.785935718 \quad (39)$$

Finally, let's use fsolve to approximate the points of intersection of the ellipse  $x^2 + 4y^2 = 25$  and the circle

$$(x - 4)^2 + (y - 4)^2 = 9:$$

> fsolve([x<sup>2</sup> + 4 · y<sup>2</sup> = 25, (x - 4)<sup>2</sup> + (y - 4)<sup>2</sup> = 9]);

$$\{x = 1.468810049, y = 2.389696479\} \quad (40)$$

OK, Maple found one of the two intersection points. To find the other one, we can specify an interval in which to search:

> fsolve([x<sup>2</sup> + 4 · y<sup>2</sup> = 25, (x - 4)<sup>2</sup> + (y - 4)<sup>2</sup> = 9], {x = 4.0 .. 4.9, y = 0.0 .. 2.0});

$$\{x = 4.538755662, y = 1.048772741\} \quad (41)$$

Note that Maple wants the list of ranges inside set braces -- {...} -- rather than inside square brackets. Unfortunately, Maple's syntax often lacks consistency.

Another function in the "solve" family is the **isolve** function, which is useful for solving Diophantine equations.

As an example, we solve Pell's equation with multiplier 61:

$$x^2 - 61y^2 = 1$$

> isolve(x<sup>2</sup> - 61 · y<sup>2</sup> = 1, t);

$$\begin{aligned}
& \left\{ x = -\frac{1}{2} (1766319049 + 226153980 \sqrt{61})^t - \frac{1}{2} (1766319049 - 226153980 \sqrt{61})^t, y = \right. \\
& \quad -\frac{1}{122} \sqrt{61} \left( (1766319049 + 226153980 \sqrt{61})^t - (1766319049 \right. \\
& \quad \left. - 226153980 \sqrt{61})^t \right) \left. \right\}, \left\{ x = -\frac{1}{2} (1766319049 + 226153980 \sqrt{61})^t \right. \\
& \quad \left. - \frac{1}{2} (1766319049 - 226153980 \sqrt{61})^t, y = \frac{1}{122} \sqrt{61} \left( (1766319049 \right. \right. \\
& \quad \left. \left. + 226153980 \sqrt{61})^t - (1766319049 - 226153980 \sqrt{61})^t \right) \right\}, \left\{ x = \frac{1}{2} (1766319049 \right. \\
& \quad \left. + 226153980 \sqrt{61})^t + \frac{1}{2} (1766319049 - 226153980 \sqrt{61})^t, y = \right. \\
& \quad \left. -\frac{1}{122} \sqrt{61} \left( (1766319049 + 226153980 \sqrt{61})^t - (1766319049 \right. \right. \\
& \quad \left. \left. - 226153980 \sqrt{61})^t \right) \right\}, \left\{ x = \frac{1}{2} (1766319049 + 226153980 \sqrt{61})^t \right. \\
& \quad \left. + \frac{1}{2} (1766319049 - 226153980 \sqrt{61})^t, y = \frac{1}{122} \sqrt{61} \left( (1766319049 \right. \right. \\
& \quad \left. \left. + 226153980 \sqrt{61})^t - (1766319049 - 226153980 \sqrt{61})^t \right) \right\}
\end{aligned} \tag{42}$$

Note that the solution is given in terms of the parameter  $t$ , which we specified as the second argument in the `isolve` command. To obtain the smallest positive solution, we evaluate the last element in the above list (of four general solutions) for  $t = 1$ :

$$\begin{aligned}
& > \text{subs}(t = 1, \%[4]); \\
& \qquad \qquad \qquad \{x = 1766319049, y = 226153980\}
\end{aligned} \tag{43}$$

A truly multipurpose function is **convert**. It has many options, but here are some simple examples, which should be self-explanatory.

$$\begin{aligned}
& > \text{convert}(43, \text{binary}); \\
& \qquad \qquad \qquad 101011
\end{aligned} \tag{44}$$

$$\begin{aligned}
& > \text{convert}(543, \text{roman}); \\
& \qquad \qquad \qquad \text{"DXLIII"}
\end{aligned} \tag{45}$$

$$\begin{aligned}
& > \text{convert}\left(\frac{\pi}{6}, \text{degrees}\right); \\
& \qquad \qquad \qquad 30 \text{ degrees}
\end{aligned} \tag{46}$$

$$\begin{aligned}
& > \text{convert}(2.625, \text{rational}); \\
& \qquad \qquad \qquad \frac{21}{8}
\end{aligned} \tag{47}$$

$$\begin{aligned}
& > \text{convert}\left(\frac{21}{8}, \text{float}\right); \\
& \qquad \qquad \qquad 2.625000000
\end{aligned} \tag{48}$$

In the context of algebra, one particularly useful option for `convert` is the **parfrac** option. This can be used to find the partial fraction decomposition of a rational function. For example:

$$\begin{aligned}
& > q := x \rightarrow \frac{4 \cdot x^4 + 5 \cdot x^3 - 25 \cdot x^2 - 5 \cdot x + 12}{(x + 2)^2 \cdot (x - 1)^3};
\end{aligned}$$

$$q := x \rightarrow \frac{4x^4 + 5x^3 - 25x^2 - 5x + 12}{(x+2)^2(x-1)^3} \quad (49)$$

> `convert(q(x), parfrac);`

$$\frac{1}{x+2} - \frac{2}{(x-1)^2} + \frac{3}{x-1} + \frac{2}{(x+2)^2} - \frac{1}{(x-1)^3} \quad (50)$$

> `simplify(%);`

$$\frac{4x^4 + 5x^3 - 25x^2 - 5x + 12}{(x+2)^2(x-1)^3} \quad (51)$$

Finally, we mention the **gcd** and **lcm** functions. These provide analogs, for polynomials, of the `igcd` and `ilcm` functions for integers.

The expression `gcd(a(x),b(x))` returns the greatest common divisor  $d(x)$  of the two polynomials  $a(x)$  and  $b(x)$ . The optional form `gcd(a(x),b(x),s,t)` will also return polynomials  $u(x)$  and  $v(x)$  such that  $a(x) = d(x)u(x)$  and  $b(x) = d(x)v(x)$

Here's an example:

> `a := x → x3 - x2 - x - 2;`

$$a := x \rightarrow x^3 - x^2 - x - 2 \quad (52)$$

> `b := x → x4 + x3 + 4 · x2 + 3 · x + 3;`

$$b := x \rightarrow x^4 + x^3 + 4x^2 + 3x + 3 \quad (53)$$

> `gcd(a(x), b(x), s, t);`

$$x^2 + x + 1 \quad (54)$$

> `s;`

$$x - 2 \quad (55)$$

> `t;`

$$x^2 + 3 \quad (56)$$

The `lcm` function takes any number of polynomial arguments and returns their least common multiple.

For example:

> `lcm(a(x), b(x));`

$$(x-2)(x^4 + x^3 + 4x^2 + 3x + 3) \quad (57)$$

> `factor(%);`

$$(x-2)(x^2 + x + 1)(x^2 + 3) \quad (58)$$

> `lcm(a(x), b(x), x2 - 4);`

$$(x^4 + x^3 + 4x^2 + 3x + 3)(x^2 - 4) \quad (59)$$

> `factor(%);`

$$(x^2 + x + 1)(x^2 + 3)(x-2)(x+2) \quad (60)$$

>