36. \( \phi((4, 4)) = \phi(4(1, 1)) = 4\phi((1, 1)) = 4\phi((3, 2) - (2, 1)) = 4(\phi((3, 2)) - \phi((2, 1)) = 4(a - b) \).

40. Define a map \( \phi : G/N \to G/M \) by \( g + N \mapsto g + M \) for all \( g \in G \). First we show \( \phi \) is well-defined. Suppose \( g_1, g_2 \in G \) such that \( g_1 + N = g_2 + N \). Then \( g_1 - g_2 \in N \). Since \( N \leq M \), \( g_1 - g_2 \in M \) also. Thus \( g_1 + M = g_2 + M \), i.e. \( \phi(g_1 + N) = \phi(g_2 + N) \).

Now let \( x \in \ker \phi \). Then \( x = g + N \) for some \( g \in G \), and \( \phi(x) = \phi(g + N) = g + M = M \). (\( M \) is the identity of \( G/M \).) It follows that \( g \in M \), i.e. \( g + N \in M/N \). We’ve shown \( \ker \phi \subseteq M/N \). For the reverse inclusion, let \( x \in M/N \). Then \( x = m + N \) for some \( m \in M \). Thus \( \phi(x) = \phi(m + N) = m + M = M \), so that \( x \in \ker \phi \). This shows \( M/N \in \ker \phi \). Thus \( M/N = \ker \phi \). Therefore, by the First Isomorphism Theorem,

\[
\frac{G/N}{M/N} \approx G/M.
\]

44. Let \( \phi : G \to G/N \) be the natural homomorphism. Let \( g \in G \). (Note \( g \) has finite order.) Then \( \phi(g) = gN \). By Property 3 of Theorem 10.1, \( |\phi(g)| \) divides \( |g| \), i.e. \( |gN| \) divides \( |g| \).

58. Let \( \phi : \mathbb{Z} \to S_3 \) be a homomorphism. Since \( \mathbb{Z} \) is cyclic, \( \phi(\mathbb{Z}) \) must be cyclic by Property 2 of Theorem 10.2. Thus \( \phi(\mathbb{Z}) \neq S_3 \) since \( S_3 \) is noncyclic. There are no homomorphisms from \( \mathbb{Z} \) onto \( S_3 \).

Since 1 is a generator of \( \mathbb{Z} \), \( \phi \) is completely determined by the image of 1. Since 1 has infinite order, there are no restrictions on the image of 1. Thus each element of \( S_3 \) is a possible image for 1. There are 6 homomorphisms from \( \mathbb{Z} \) to \( S_3 \).