

## Antiderivatives!

Are you anti-derivative? (Might be after that last test, eh?....)

Seriously, here “anti” means *opposite*—We’ll be going in the opposite direction, from the derivative back to the original function. (It’s kind of like the show *Jeopardy*.)

**Alex Trebek:**  $x^2$

**Contestant:** What is the derivative of  $\frac{1}{3}x^3 + C$ , where  $C$  is any constant?

**Question.** Given a function, of what is it the derivative?

**Definition 1** *Let  $f$  be a function. If  $F$  is another function such that  $F' = f$  on an interval  $I$ , then  $F$  is called an antiderivative of  $f$  on  $I$ .*

Every function that has *one* antiderivative has *INFINITELY MANY!* (But they are all closely related.)

**Example 1** *Consider  $f(x) = x^2$ .*

ANTIDERIVATIVE:  $F(x) = \frac{1}{3}x^3$ . (Check this by computing  $F'$ .)

But  $G(x) = \frac{1}{3}x^3 + 1$  is also an antiderivative of  $f$ . The derivative of 1 is zero, so  $G$  has the same derivative as  $F$ , namely  $f$ . Similarly, we can add any constant onto  $F$  and get another antiderivative of  $f$ . In this way we get a *family* (set) of antiderivatives for  $f(x) = x^2$  given by  $F(x) = \frac{1}{3}x^3 + C$  for any real number  $C$ .

The same thing works in general, giving a family of curves which all look the same, but are shifted vertically. (See Figure 1 on p. 340 of the text for some graphs of the family above.) This makes sense if you consider that the derivative tells us the *shape* of a curve, not its location.

We get the following characterization of all the antiderivatives of a function.

**Theorem 1** *Suppose  $F$  is an antiderivative of  $f$  on an interval  $I$ . Then another function  $G$  is an antiderivative of  $f$  if and only if  $G(x) = F(x) + C$  for some  $C \in \mathbb{R}$ . Thus*

$$\{\text{antiderivatives of } f\} = \{F(x) + C \mid C \in \mathbb{R}\}.$$

IMPORTANT: In order to get the most general antiderivative of a function, *always add a constant  $C$*  to any one antiderivative.

**Note:** The comments above should convince you that  $F(x) + C$  is an antiderivative if  $F(x)$  is. But what about the other direction of this Theorem? How do we know that *every* antiderivative has this form? It follows from a theorem called the *Mean Value Theorem* (from Section 4.2, which we won't cover) that any two functions that have the same derivative, *i.e.*, any two antiderivatives of the same function, must differ by a constant  $C$ .

**Example 2** For the function  $f$  given by each of the following formulas, give

a) **an antiderivative**  $F$

b) **the most general antiderivative**  $G$

1.  $f(x) = \cos x$

2.  $f(x) = 1/x$

3.  $f(x) = 1/x^2$

4.  $f(x) = x^n$  for  $n \neq -1$

Go to the next page for the answers.

For (1), we choose  $F(x) = \sin x$  since  $\frac{d}{dx}[\sin x] = \cos x$ . Thus the most general antiderivative is given by  $G(x) = \sin x + C$ .

For (2), we can choose  $F(x) = \ln x$  as an antiderivative since we know  $\frac{d}{dx}[\ln x] = 1/x$ . This time the most general antiderivative is given by  $G(x) = \ln |x| + C$ . Why do we need the absolute values? The function given by  $F(x) = \ln x$  is only defined on the interval  $(0, \infty)$ , so it is only an antiderivative for  $f(x) = 1/x$  on this limited interval. However, the function given by  $\ln |x|$  also has derivative  $1/x$ , and it is defined on the whole real line except at 0. (This is the best we can hope for in this case.) Hence this a more general antiderivative since it is valid on a larger domain. In general, the most general antiderivative will be the one valid on the largest subset of  $\mathbb{R}$ .

For (3), we try to think of a function whose derivative  $f$  is given by  $f(x) = 1/x^2$ . Well, the derivative of  $1/x$  is  $-1/x^2$ . That's only off by a minus sign. We can easily fix this by sticking a minus sign on the  $1/x$ . Then  $\frac{d}{dx}[-1/x] = -\frac{d}{dx}[1/x] = -(-1/x^2) = 1/x^2$ . So  $F(x) = -1/x$  is an antiderivative, and the most general antiderivative is given by  $G(x) = -1/x + C$ . Constant multiples always pull out of derivatives due to the Constant Multiple Rule, so if we can find a function whose derivative is off by a constant multiple from the desired function, just multiply that function by the necessary constant (the reciprocal of the constant multiple above) to get your antiderivative.

Part (4) tries to generalize from specific cases like in (3) to a kind of backward version of the Power Rule for derivatives. Remember the Power Rule says that  $\frac{d}{dx}[x^n] = nx^{n-1}$ . Thus, to find a function whose *derivative* is  $x^n$ , we need to do the opposite of what we did in the Power Rule: Well,  $x^{n+1}$  has derivative  $(n+1)x^n$ . We just divide by that pesky factor of  $n+1$  to get our antiderivative

$$F(x) = \frac{1}{n+1}x^{n+1}.$$

Notice that

- instead of *subtracting* 1 from the power, we *add* 1.
- instead of *multiplying* by the *old* power, we *divide* by the *new* power.

See, it's exactly the opposite of the Power Rule. Is this the most general antiderivative?...

NOPE! It's  $G(x) = \frac{1}{n+1}x^{n+1} + C$ . Don't forget to add the constant!

**Note:** This formula doesn't work when  $n = -1$  because then we'd be dividing by 0. Remember  $f(x) = x^{-1}$  is a special case already covered in Part 2 of the example.

We can use the same procedure for reversing all our basic rules for derivatives. We get the antidifferentiation formulas found in Table 1, which is essentially a table of basic derivatives if read backwards. In this table, suppose  $F' = f$  and  $G' = g$ .

Function	<u>An</u> antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n$ ( $n \neq 1$ )	$\frac{x^{n+1}}{n+1}$
$1/x$	$\ln x $
$e^x$	$e^x$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$

Table 1: Table of Antidifferentiation Formulas

You can also find these formulas on page 341 in the text. Notice these are not the most general forms of the antiderivatives. You'd need to add a  $C$  to each one to get the general forms.

Don't imagine that you know how to antidifferentiate every function now. Consider an example like  $xe^x$ . It's harder—this kind of thing is covered in Calculus II. The only way we can hope to find the antiderivative of a product or quotient function at this point is to try to rewrite it in a simpler form so that the above rules apply. In fact, there are functions are impossible to antidifferentiate using our current approach. Quite a bit of imagination is required to find some other antiderivatives. This is where math begins to look like the art form that it truly is, rather than a mechanical process.

**Example 3** Find all functions  $g$  such that

$$g'(x) = 4 \sin x - 3x^5 + 6\sqrt[4]{x^3}.$$

This is a very simple example of a *differential equation*, as are all the antiderivative problems you will encounter in this section. Try it on your own. See the last page for a solution.

**Definition 2** An equation that involves derivatives is called a differential equation.

There are whole courses about differential equations; they are also sometimes covered briefly in University Calculus II.

Want to find an antiderivative that is a *specific function*? You can if you can nail it down at one point (*i.e.* know its value at that point).

**Example 4** Find  $f$  if  $f'(x) = e^x + \frac{20}{1+x^2}$  and  $f(0) = -2$ .

See the next page for the solution.

Finally, we can look at word problems involving antiderivatives.

**Example 5** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground  $t$  seconds later. When does it reach its maximum height? When does it hit the ground?

In order to solve this, you'll need to know that acceleration due to gravity is 9.8 m/s or, equivalently, 32 ft/s *downward*. See the next page for a solution.

**Homework Assignment—pp.345-346: 1-7 odd, 11, 13, 15, 21, 25-49 odd, 57-63 odd**

**On WebAssign, WA#21 covers this material and is due at 11:00pm on Thursday, December 9.**

**And, yes, this material *is* fair game for the final exam!**

**Solution to antiderivative problem:**  $g(x) = -4 \cos x - \frac{1}{2}x^6 + \frac{24}{7}x^{7/4} + C$ . You can check by taking its derivative.

**Solution to the differential equation (2nd antiderivative problem):**

$$f(x) = e^x + 20 \tan^{-1} x - 3.$$

**Hint:** First find the general antiderivative  $f(x) = e^x + 20 \tan^{-1} x + C$ . Set  $f(0) = -2$  (so that  $-2 = e^0 + 20 \tan^{-1} 0 + C = 1 + 0 + C$ ) and solve for  $C$ .

**Solution to word problem:** This problem too involves a differential equation. The second derivative of the height function is the function with constant value  $-32$ . (The acceleration is downward.) It is the derivative of the velocity function  $v$  of the form  $v(t) = -32t + C$ . But this is the derivative of the height function, which thus has form  $h(t) = -16t^2 + Ct + D$ , where  $D$  is another constant. (We are antidifferentiating here.) But  $C = v(0) = 48$  is the initial velocity, and  $D = h(0) = 432$  is the initial height. So the height function after  $t$  seconds is given by  $h(t) = -16t^2 + 48t + 432$ . From here you should know how to answer the two remaining questions. I get 1.5 seconds for the first and approximately 6.9 seconds for the second.