

# WHY IS $\mathcal{O}(1)$ CALLED THE TWISTING SHEAF?: THE BUNDLES $\mathcal{O}(n)$ ON PROJECTIVE SPACE

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## 1. INTRODUCTION

**Question:** Why is  $\mathcal{O}(1)$  called the twisting sheaf?

For simplicity we specialize to  $\mathbb{P}_{\mathbb{C}}^1$  during most of the talk. However, most of what is said could easily be generalized to  $\mathbb{P}_{\mathbb{C}}^r$  and probably to arbitrary complex projective varieties.

**Answer:**

$$\begin{aligned}\mathcal{O}(1) &= \widetilde{S(1)} \\ &\leftrightarrow \{\text{bundle on } \mathbb{P}^1 \text{ with transition functions } g_{10} = x_0/x_1, g_{01} = x_1/x_0\} \\ &= \text{universal bundle on } \mathbb{P}^1 \\ &= \text{trivial bundle on } \mathbb{P}^1 \text{ with one twist put in}\end{aligned}$$

Goals:

- (1) Briefly introduce the language of bundles
- (2) Describe twisting geometrically
- (3) Show  $\mathcal{O}(1)$  is also the universal bundle
- (4) Show greater twisting gives higher powers in transition functions
- (5) (Introduce Chern classes if time permits)

**Definition 1.** Let  $S$  be a graded ring,  $M$  a graded  $S$ -module. For any  $n \in \mathbb{Z}$ , we define the twisted module  $M(n)$  by shifting  $n$  places to the left, i.e.  $M(n)_d = M_{d+n}$ .

(Draw picture.)

**Definition 2.** Let  $X = \text{Proj}(S)$ . For  $n \in \mathbb{Z}$ , define  $\mathcal{O}_X(n)$  to be  $\widetilde{S(n)}$ . If  $\mathcal{F}$  is any sheaf of  $\mathcal{O}_X$ -modules, let  $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ .

Let  $A$  be a ring.

**Question:** What is  $\Gamma(\mathbb{P}_{\mathbb{A}^r}^r, \mathcal{O}(n))$ ?

**Proposition 1.** Global sections of  $\mathcal{O}_{\mathbb{P}_{\mathbb{A}^r}^r}(n)$  are degree  $n$  polynomials in  $r + 1$  variables.

**Pf.**

In this case  $S = A[x_0, \dots, x_r]$ .

$$\Gamma(\mathbb{P}^r, \mathcal{O}(n)) \stackrel{\text{def}}{=} \Gamma(\mathbb{P}^r, \widetilde{S(n)}) \stackrel{5.12(b)}{\cong} \Gamma(\mathbb{P}^r, \widetilde{S}(n)) \stackrel{\text{def}}{=} (\Gamma_*(\widetilde{S}))_n \stackrel{5.13}{\cong} S_n.$$

The last item is the module of degree  $n$  polynomials over  $A$ . Results referenced are in [4, Ch. II].  $\square$

## 2. VECTOR BUNDLES

In order to “see” the twisting, we need to look at the geometric analogue of sheaves, which are algebraic.

Let  $k$  be a field.

**Definition 3.** A continuous map  $\pi : E \rightarrow X$  of Hausdorff spaces is called a  $k$ -vector bundle of rank  $r$  if the following conditions are satisfied:

- (1)  $E_p := \pi^{-1}(p)$  (the fiber over  $p$ ) is a  $k$ -vector space of dimension  $r$  for all  $p \in X$ .
- (2) For every  $p \in X$  there exists a neighborhood  $U$  of  $p$  and a homeomorphism  $h : \pi^{-1}(U) \rightarrow U \times k^r$  such that  $h(E_p) \subset \{p\} \times k^r$  ( $h$  preserves fibers.) Furthermore  $h^p : E_p \xrightarrow{h} \{p\} \times k^r \xrightarrow{\pi_2} k^r$  must be a  $k$ -vector space isomorphism for all  $p$ .

The pair  $(U, h)$  is called a *local trivialization*.

**Example 1.** A bundle of the form  $k^r \times X \xrightarrow{\pi_2} X$  is called a *trivial bundle*. We can also say this bundle “has no twisting.”

**2.1. Transition functions.** Suppose we have two local trivializations  $(U_\alpha, h_\alpha)$  and  $(U_\beta, h_\beta)$  of a bundle  $E \rightarrow X$ . We can compose to get  $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times k^r \rightarrow (U_\alpha \cap U_\beta) \times k^r$ . This induces a map  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, k)$  given by  $g_{\alpha\beta}(p) = h_\alpha^p \circ (h_\beta^p)^{-1} : k^r \rightarrow k^r$ .

The functions  $g_{\alpha\beta}$  are called the *transition functions* of the bundle  $E \rightarrow X$ . The transition functions satisfy the following compatibility conditions:

- (1)  $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = I_r$  on  $U_\alpha \cap U_\beta \cap U_\gamma$  (the cocycle condition).
- (2)  $g_{\alpha\alpha} = I_r$  on  $U_\alpha$  (the identity condition).

**Definition 4.** A section of a bundle  $E \rightarrow X$  is a morphism  $s : X \rightarrow E$  such that  $\pi \circ s = \text{Id}_X$ . ( $s$  maps all points into their fibers.) The set of sections of  $E|_U$  over  $U \subset X$  is denoted  $\Gamma(U, E)$ .

Facts about sections:

- (1)  $\Gamma(U, E)$  is actually a  $k$ -vector space and an  $\mathcal{O}_U$ -module.

- (2) As  $U$  varies over all open sets in  $X$ , the  $\Gamma(U, E)$  fit together to form a sheaf  $\mathcal{E}$  of  $\mathcal{O}_X$ -modules, called the sheaf of sections of  $E$ . It is a locally free sheaf of rank  $r$ .
- (3) We have a contravariant isomorphism between the categories

$$\left\{ \begin{array}{c} \text{vector bundles} \\ \text{on } X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{locally free} \\ \text{sheaves on } X \end{array} \right\}$$

given by  $E \mapsto \mathcal{E}$ . The inverse is something like  $\mathcal{E} \mapsto \underline{\text{Spec}}(\text{Sym}(\mathcal{E}^\vee))$ .

**Conclusion:** Locally free sheaves and vector bundles are “the same thing.”

**2.2. Alternate definition of line bundles.** We can also *define* bundles in terms of transition functions. For line bundles, this just involves giving an open cover  $\{U_\alpha\}$  and a collection of nowhere vanishing holomorphic functions  $g_{\alpha\beta}$  on each  $U_\alpha \cap U_\beta$  satisfying the cocycle and identity conditions. These functions can be used to construct a unique line bundle in the previous sense, so the definitions are equivalent. While not very geometric, this definition is more useful for some purposes. I am sweeping many details under the rug.

The (equivalent) definition of a section also looks different in this case:

**Definition 5.** *Let  $L$  be a line bundle with transition functions  $g_{\alpha\beta}$  on an open cover  $\{U_\alpha\}$ . A holomorphic section of  $L$  is a collection of holomorphic functions  $s_\alpha$  on each  $U_\alpha$  satisfying  $s_\alpha = g_{\alpha\beta}s_\beta$  on each  $U_\alpha \cap U_\beta$ .*

### 2.3. Generalization and key examples.

**Example 2** (Universal bundle). *The most important example for us is the universal bundle  $\mathcal{U}(k, n) \rightarrow G(k, n)$  over a Grassmannian, which parametrizes  $k$ -dimensional subspaces of an  $n$ -dimensional vector space. In particular,  $\mathbb{P}^r = G(1, r+1)$ , so  $\mathbb{P}^r$  has a universal bundle. It seemed that Josh was just about to get to this at the end of his talk last week. It satisfies a nice universal property related to the functor he mentioned, but I won't have time to go into that. For our purposes we view it as follows:*

*(Draw picture)*

$\mathcal{U}_{k,n}$  is the subbundle of the trivial bundle  $G(k, n) \times \mathbb{A}^n$  with the property that the fiber over each point in  $G(k, n)$  is the line in  $\mathbb{A}^n$  represented by that point. (The universal bundle is also sometimes called the tautological bundle.)

**Example 3.** Let  $\mathcal{U} = \mathcal{U}_{1,2}$  be the universal bundle over  $\mathbb{P}^1$ .  $\mathcal{U}$  has fiber over  $[a : b]$  equal to the line in  $\mathbb{A}^2$  spanned by  $(a, b)$ . The biggest step in answering our question for  $\mathbb{P}^1$  will be to show that  $\mathcal{U} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ .

Bundles can be defined in much more generality. These are called fiber bundles. These are similar to vector bundles in many ways, but the fibers don't have to be vector spaces.

**Example 4.**  $S^1 \times [-1, 1] \rightarrow S^1$  is a trivial bundle. Draw picture.

**Example 5.** Möbius strip to central circle. Not trivial. Draw picture.

A nice illustration. But what does this have to do with  $\mathcal{O}(1)$ ? We can get to it in two steps. Extend fibers from  $[-1, 1]$  to  $\mathbb{R}$ . Call this vector bundle  $E$ . Note that  $S^1 = \mathbb{RP}^1$ , the space of lines in  $\mathbb{R}^2$ . Draw picture.  $E$  is the universal bundle for  $\mathbb{RP}^1$ . Now complexify. This is harder to visualize, but certainly the same sort of geometry is going on.

### 3. TRANSITION FUNCTIONS OF THE UNIVERSAL BUNDLE ON $\mathbb{P}^1$

Let's express  $\mathcal{U}$  in terms of transition functions.

Let  $U_i = \{(x_0 : x_1) | x_i \neq 0\}$  for  $i \in \underline{2}$  be the standard open cover of  $\mathbb{P}^1$ . We can trivialize  $\mathcal{U}$  over this cover. Let  $\pi : \mathcal{U} \rightarrow \mathbb{P}^1$  be the bundle map. A point in the fiber over  $(1 : x_1)$  has the form  $((1 : x_1), (\lambda, \lambda x_1))$  for some  $\lambda \in \mathbb{C}$ , considering  $\mathcal{U}$  as a subbundle of  $\mathbb{P}^1 \times \mathbb{A}^2$ . We define a trivialization  $h_0 : \pi^{-1}(U_0) \rightarrow U_0 \times \mathbb{A}^1$  by  $h_0((1 : x_1), (\lambda, \lambda x_1)) = ((1 : x_1), \lambda)$ . This map is obviously an homeomorphism. Similarly, we define  $h_1 : \pi^{-1}(U_1) \rightarrow U_1 \times \mathbb{A}^1$  by  $h_1((x_0 : 1), (\lambda x_0, \lambda)) = ((x_0 : 1), \lambda)$ .

**Question:** Why can't we just trivialize  $\mathcal{U}$  on all of  $\mathbb{P}^1$ ?

Now we can get the transition functions. Consider the map  $h_1 \circ h_0^{-1} : U_0 \times \mathbb{A}^1 \rightarrow U_1 \times \mathbb{A}^1$ . Really we have to restrict this map to  $(U_0 \cap U_1) \times \mathbb{A}^1$  so both  $h_0$  and  $h_1$  make sense. Thus we can divide by both  $x_0$  and  $x_1$ . Let  $(x_0 : x_1) \in (U_0 \cap U_1)$ . Rewrite it as  $(1 : x_1/x_0)$ .  $h_0^{-1}$  takes  $((1 : x_1/x_0), \lambda)$  to  $((1 : x_1/x_0), (\lambda, \lambda x_1/x_0))$ . In order to apply  $h_1$ , we rewrite this as  $((x_0/x_1 : 1), ((\lambda x_1/x_0)x_0/x_1, \lambda x_1/x_0))$ . Then  $h_1$  takes this point to  $((x_0/x_1 : 1), \lambda x_1/x_0)$ . We can now define the transition function  $g_{01} : U_0 \cap U_1 \rightarrow \mathbb{A}^1$ . Let  $p = (x_0 : x_1) \in U_0 \cap U_1$ . Then  $g_{01}(p) = h_1^p \circ (h_0^p)^{-1} = f : \mathbb{C} \rightarrow \mathbb{C}$ , where  $f(\lambda) = x_1/x_0 \lambda$ . A more compact way to write this is  $g_{01} = x_1/x_0$ . Similarly,  $g_{10} = x_0/x_1$ .

Next we describe the sheaf of sections of  $\mathcal{U}$ . For our purposes, it will suffice to find the global sections and the sections over the affine open sets  $U_0$  and  $U_1$ . Let  $z$  be an affine coordinate on  $U_0$  and  $w$  an affine coordinate on  $U_1$ . In practice, this means  $z = x_1/x_0$  and  $w = x_0/x_1$ . Thus on  $U_0 \cap U_1$  we have the identifications  $w = 1/z$  and  $z = 1/w$ .

Furthermore, this says  $g_{01} = z$  and  $g_{10} = w$  are the transition functions for the bundle  $\mathcal{U}$  with respect to the standard cover.

A global section has the form  $s = (s_0, s_1)$ , where the  $s_i : U_i \rightarrow \mathbb{C}$  are holomorphic and satisfy  $s_i = g_{ij}s_j$ . In our case it suffices to check one such identity since the transition functions are inverse to each other.

The homogeneous coordinates  $x_0, x_1$  on  $\mathbb{P}^1$  are locally projections to  $\mathbb{C}$ .  $x_0$  is the first projection: we can write  $(x_0)_0 : (1 : z) \mapsto 1$  on  $U_0$  and  $(x_0)_1 : (w : 1) \mapsto w$  on  $U_1$ . Similarly,  $(x_1)_0 : (1 : z) \mapsto z$  on  $U_0$  and  $(x_1)_1 : (w : 1) \mapsto 1$  on  $U_1$ . Notice that both  $x_0$  and  $x_1$  satisfy the requirement  $(x_k)_0 = g_{01}(x_k)_1$  for a section since  $1 = zw$  and  $z = z1$ . Thus we have found two sections of  $\mathcal{U}$ . Clearly these sections are independent. (Suppose  $ax_0 + bx_1 = 0$ . Apply this expression to  $(1 : 0)$  and  $(0 : 1)$ .) I claim that these two span the vector space of global sections. Let  $s = (f_0(z), f_1(w))$  be any section.  $f_0$  and  $f_1$  must be holomorphic, which in our context means they have to be polynomials. They also must satisfy  $f_0(z) = zf_1(w)$ . It follows that  $\deg f_0 \leq 1$  and  $\deg f_1 \leq 1$ . Finally, one can check that their degrees have to be different. So the space of sections is isomorphic to the space of linear polynomials in one variable. This space has dimension two.

Since the sets  $U_i$  give a trivialization of the bundle  $\mathcal{U}$ , the sections on  $U_i$  are just the holomorphic functions on  $U_i$ . The sections above also restrict to sections on  $U_0$ . Furthermore, on  $U_0$  we have independent sections  $x_1^{k+1}/x_0^k$  for every  $k \in \mathbb{N}$  since we are allowed to divide by  $x_0$ . These sections should span  $\Gamma(U_0, \mathcal{U})$ . Similarly  $x_0$  and  $x_1$ , together with  $x_0^{k+1}/x_1^k$  for  $k \in \mathbb{N}$ , span  $\Gamma(U_1, \mathcal{U})$ .

Notice that on these three open sets, the sections coincide exactly with those of  $\mathcal{O}(1) = \widetilde{S}(1)$ . For the global sections, this follows from our initial algebraic proposition. Then using [4, Prop 5.11(b)], and [4, Prop 5.13], we can see, for example, that

$$\begin{aligned} \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1}(1)) &= \Gamma(D_+(x_0), \widetilde{S}(1)) \\ &= \Gamma(D_+(x_0), (\widetilde{S(1)}|_{D_+(x_0)})) = \Gamma(D_+(x_0), (\widetilde{S(1)}_{(x_0)})) = S(1)_{(x_0)}. \end{aligned}$$

Now for two sheafs to officially be equal, they should have the same space of sections over *every* open set. Our purposes only require that they be isomorphic. But it turns out that equality of the sections on the two affine sets is enough to give us this result! We have two sheaves whose sections are equal on an affine cover. We need only be convinced that two sheaves on an affine space with the same global sections are isomorphic. Then the result probably follows from uniqueness of a sheaf obtained from glueing. We may be relying on that fact that all

sections are contained in the same function field.) But  $\mathcal{O}(1)$  is coherent by definition, and  $\mathcal{U}$  is locally free, hence coherent. Thus we may apply [4, Cor 5.5], which says that taking global sections gives an equivalence of categories in this case.

**Bad news/Warning:** Some people say  $\mathcal{O}(-1)$  is the universal bundle. This has to do with the dueling dual perspectives in algebraic geometry that Josh mentioned last week, *e.g.* whether we consider projective space to parametrize lines or linear functionals on affine space, etc.

#### 4. MORE TWISTING

$\mathcal{O}_{\mathbb{P}^1}(2)$  has global sections generated by  $x_0^2$ ,  $x_0x_1$ , and  $x_1^2$ . We can determine transition functions to be

$$g_{01} = z^2 \text{ and } g_{10} = w^2.$$

We can generalize this easily to see that  $\mathcal{O}(n)$  has transition functions  $g_{01} = z^n$  and  $g_{10} = w^n$ .

**Pattern:** Higher twisting on  $\mathbb{P}^r$  means higher powers in the transition functions.

Notice that a section like  $x_0^n$  has  $n$  zeros. (And no poles!) As a result, counting the zeros of a section can be used to measure the twisting of a bundle on  $\mathbb{P}^1$ . This leads to Chern classes. In general (sufficiently nice) situations, the zero set of a section gives the *top Chern class*.

#### 5. CHERN CLASSES

The Chern classes of a bundle measure how far a bundle is from being trivial, *i.e.* how twisted it is. Suppose  $X$  is a variety or scheme of dimension  $n$ . Then for  $i \in \mathbb{N} \cup \{0\}$  we have operations

$$c_i : (\text{Bundles on } X) \rightarrow A^{n-i}(X) = H^{2n-2i}(X)$$

Facts about Chern classes:

- (1)  $c_0(E) = 1$  for all bundles  $E \rightarrow X$ .
- (2)  $c_i(E) = 0$  for  $i > \text{rank}(E)$  and  $i > \dim(X)$ .
- (3) If  $L$  and  $L'$  are line bundles on  $X$ ,
  - (a)  $c_1(L \otimes L') = c_1(L) + c_1(L')$  and
  - (b)  $c_1(L^\vee) = -c_1(L)$ .

Facts (1) and (2) tell us that for line bundles  $c_1$  is the only interesting Chern class. On  $\mathbb{P}^r$  we have a group homomorphism

$$c_1 : \text{Pic}(\mathbb{P}^r) \rightarrow \text{Div}(X) \simeq \mathbb{Z},$$

where the isomorphism on the right takes  $n[H]$  to  $n$ . In fact,  $c_1$  is an isomorphism as well in this case, so that the group of isomorphism classes of line bundles on  $\mathbb{P}^r$  is isomorphic to  $\mathbb{Z}$ . We can at least see that it is an injective homomorphism easily from the facts above. We have:

$c_1(\mathcal{O}_{\mathbb{P}^r}) = 0$ . This has to be true. Plus it makes sense since the structure bundle has no twisting.

$c_1(\mathcal{O}_{\mathbb{P}^r}(1)) = 1$ . This we will have to take on faith. But intuitively it makes sense if we think of  $\mathcal{O}_{\mathbb{P}^r}(1)$  as having one twist.

$$c_1(\mathcal{O}_{\mathbb{P}^r}(n)) = c_1(\mathcal{O}_{\mathbb{P}^r}(1)^{\otimes n}) = nc_1(\mathcal{O}_{\mathbb{P}^r}(1)) = n \text{ for } n > 0.$$

$$c_1(\mathcal{O}_{\mathbb{P}^r}(-1)) = c_1(\mathcal{O}_{\mathbb{P}^r}(1)^\vee) = -c_1(\mathcal{O}_{\mathbb{P}^r}(1)) = -1.$$

Thus  $c_1(\mathcal{O}_{\mathbb{P}^r}(n)) = n$  for  $n < 0$  as well.

## 6. FINAL REMARKS

References for the analytic approach to algebraic geometry include [3] and [7]. The heavy analytical emphasis of the former can be quite intimidating. You may want to skip many parts of Chapter 0 on a first reading. My definitions for vector bundles were taken straight from [7].

The alternate definition of vector bundles comes from [5]. I highly recommend these lecture notes as an introduction to the geometric side of algebraic geometry without too much technical machinery. It also shows how this stuff connects to physics. Some of the later lectures are evidently not complete, and Lecture 14 doesn't exist at all yet. Most can be gotten from Sheldon's OSU web site, but this site may not exist for much longer. Starting with Lecture 11, they are available on the referenced UIUC pages.

A development of bundles in the more general sense can be found in [6]. There are probably some more recent treatments as well.

The theory of Chern classes is fully developed in [2]. [1] is another great source for analytic type stuff that can be used very powerfully in algebraic geometry.

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