

# Research plan

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The two topics detailed below for current and future research are uniformly primary ideals and the intersection theory of moduli spaces of stable maps.

**Uniformly primary ideals.** A *primary ideal*  $I$  of a commutative ring  $R$  may be defined as an ideal for which, given any  $a, b \in R$  with  $ab \in I$  and  $a \notin I$ , there exists a natural number  $n$  such that  $b^n \in I$ . However,  $n$  is not bounded above in general. In joint work with A. Hetzel, I am considering his more restrictive definition of a *uniformly primary ideal*.

**Definition 1** *Let  $R$  be a commutative ring with  $1 \neq 0$ . An ideal  $I$  of  $R$  is uniformly primary if there exists  $N \in \mathbb{N}$  such that whenever  $ab \in I$  for  $a, b \in R$  and  $a \notin I$ , then  $b^N \in I$ . A uniformly primary ideal has order  $N$  if  $N$  is the smallest natural number for which this property holds.*

Prime ideals are precisely the uniformly primary ideals of order 1. Moreover, we expect that in non-Noetherian rings the behavior of a uniformly primary ideal of *any* order is significantly closer to that of a prime ideal than is the behavior of an arbitrary primary ideal. In Noetherian rings the notions coincide.

**Proposition 1** *Let  $R$  be a Noetherian ring. Then an ideal is primary if and only if it is uniformly primary for some  $N$ .*

Although we don't get a new class of ideals in this case, knowing the order of a primary ideal still proves useful. In particular, the order gives geometric information about the infinitesimal neighborhoods of the corresponding affine subscheme. Here is one case.

**Proposition 2** *Let  $k$  be an algebraically closed field (of characteristic 0),  $R = k[x_1, \dots, x_n]$ , and  $I = (x_1^{d_1}, \dots, x_n^{d_n})$  a uniformly primary ideal of order  $M$ . Also let  $X = V(I) \subset \text{Spec } R$ . Then  $M$  is the smallest natural number such that for all  $(j_1, \dots, j_n)$  with  $\sum j_i = M$ , there exists  $f \in I$  such that  $\frac{\partial^M f(0)}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} \neq 0$ .*

In other words, the  $M$ 'th-order infinitesimal neighborhood of the origin is the lowest for which  $V(I)$  contains no information.

We have proved several significant results about uniformly primary ideals, and a paper describing our work is currently in preparation. We are also continuing to consider various conjectures regarding uniformly primary ideals including the following.

**Conjecture 1** *Let  $R$  be a Noetherian ring and  $P$  a prime ideal in  $R$  that can be generated by a regular sequence, and  $M \in \mathbb{N}$ . Then  $P^M$  is uniformly primary of order  $M$ .*

**Conjecture 2** *The ring extensions that preserve the order of a uniformly primary ideal are precisely the algebraic extensions.*

**Intersection theory of moduli spaces of stable maps.** There has been much interest recently among mathematicians and physicists in the moduli spaces  $\overline{M}_{g,n}(\mathbb{P}^r, d)$  of degree  $d$  stable maps from  $n$ -pointed, genus  $g$  curves to projective space (and more generally in the spaces  $\overline{M}_{g,n}(X, \beta)$  of stable maps to a variety  $X$  representing a homology class  $\beta$ ), first defined in [5]. Gromov-Witten invariants can be defined as intersection numbers in moduli spaces of stable maps. These invariants are closely related to correlation functions in the topological sigma model coupled to gravity, which is a particular string theory. String theory in general has great potential to be part of a fundamental physical description of the universe. These moduli spaces have also been used to give answers to many problems of enumerative geometry that were inaccessible by previous methods.

At first, techniques were developed which allowed intersection theory to be carried out in top codimension in arbitrary moduli spaces of stable maps using indirect methods such as localization under torus actions [5]. It was acknowledged that presentations for the Chow rings of  $\overline{M}_{g,n}(X, \beta)$  would give a much better understanding of their geometry and would allow unfettered calculation of intersections in these spaces. Until 2002, such presentations were known in only a few special cases where the moduli spaces correspond to more familiar algebro-geometric objects. Then Behrend and O'Halloran showed that over  $\mathbb{C}$  the cohomology ring of  $\overline{M}_{0,0}(\mathbb{P}^r, d)$  stabilizes as  $r$  becomes large. They denoted the limiting ring by  $H^*(\overline{M}_{0,0}(\mathbb{P}^\infty, d), \mathbb{C})$  and gave a presentation for the ring  $H^*(\overline{M}_{0,0}(\mathbb{P}^\infty, 3), \mathbb{C})$  in [1]. They also gave presentations for the rings  $H^*(\overline{M}_{0,0}(\mathbb{P}^r, 2), \mathbb{C})$ . The following year, Mustata and Mustata described the Chow rings  $\overline{M}_{0,1}(\mathbb{P}^r, d)$  in [6]. My dissertation in 2004

gave the first presentation for a Chow ring of a moduli space of stable maps with degree greater than one and more than one marked point ([3],[2]). In particular, I found a presentation for  $A^*(\overline{M}_{0,2}(\mathbb{P}^1, 2))$ . In addition, I later described an additive basis for the more general class of Chow rings of the spaces  $\overline{M}_{0,2}(\mathbb{P}^r, 2)$ . Then in February 2005 Getzler and Pandharipande ([4]) announced their calculation of the Betti numbers for all the moduli spaces  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  of stable maps from genus 0 curves. Finally, in July 2005 Mustata and Mustata described presentations for the Chow rings of these spaces in [7]. However, their work was based on a very complicated construction of a chain of *intermediate moduli spaces* and on the notion of an *extended Chow ring*. In the future, I would like to find presentations for the Chow rings of more moduli spaces of stable maps using the more direct methods employed in my dissertation, which should give a simpler and more geometric understanding of the relations in these Chow rings. If time permits, I might also investigate presentations for Chow rings of moduli spaces of stable maps in the open cases of positive genus curves and more general target spaces  $X$ .

## References

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