Chapter Three
Determinants

3.1 INTRODUCTION

Consider again a homogeneous system of $n$ linear equations in the $n$ variables $x_1, x_2, \ldots, x_n$:

$$
\begin{align*}
  c_{11}x_1 &+ c_{12}x_2 + \cdots + c_{1n}x_n = b_1 \\
  c_{21}x_1 &+ c_{22}x_2 + \cdots + c_{2n}x_n = b_2 \\
  &\vdots \\
  c_{m1}x_1 &+ c_{m2}x_2 + \cdots + c_{mn}x_n = b_m
\end{align*}
$$

As we know, the solution set of the system $H$ is completely determined by its coefficient matrix

$$
C = \begin{bmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{21} & c_{22} & \cdots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} & c_{n2} & \cdots & c_{nn}
\end{bmatrix}
$$

In fact, we have the following important result from the preceding chapter, which we restate here for convenience.

**Theorem 2.5:** For a given $n$ by $n$ matrix $C$, the following statements are equivalent:

1. $C$ is invertible.
2. The homogeneous system of $n$ linear equations in $n$ variables $CX = Z$ has only the trivial solution.
3. $C$ is row equivalent to the $n$ by $n$ identity matrix $I$.
4. There exists a sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$
E_k \cdots E_2 E_1 C = I
$$

□
What we want to do in this chapter is associate a number \(\det(C)\) with an \(n\) by \(n\) matrix \(C\); this number is called the \textit{determinant} of \(C\). We want to define \(\det(C)\) so that it will have the following property:

\[
\text{\(C\) is invertible if and only if \(\det(C) \neq 0\)}
\]

We will then be able to extend Theorem 2.5 by adding a fifth equivalent condition concerning the matrix \(C\) — namely, that \(\det(C) \neq 0\).

Equivalently (in view of Theorem 2.5), the number \(\det(C)\) should be defined so that:

The homogeneous system \(CX = Z\) has a nontrivial solution if and only if \(\det(C) = 0\)

To get an idea how we might define the determinant function in general, let’s consider the particular cases \(n = 1, n = 2,\) and \(n = 3\).

**Example 3.1:** Consider the case \(n = 1\). In this case we have one equation in one variable:

\[
c_{11}x_1 = 0
\]

Clearly, this equation has a nontrivial solution if and only if \(c_{11} = 0\). Thus, we should define the determinant of the 1 by 1 matrix \([c_{11}]\) to be the number \(c_{11}\).

For the case \(n = 2\), we have a homogeneous system of two equations in two variables:

\[
\begin{align*}
c_{11}x_1 + c_{12}x_2 &= 0 \\
c_{21}x_1 + c_{22}x_2 &= 0
\end{align*}
\]

Assume, without loss of generality, that \(c_{22} \neq 0\). Employing the method of substitution, let’s multiply the first equation by \(c_{22}\), solve for \(c_{22}x_2\) in the second equation, and then substitute the resulting expression for \(c_{22}x_2\) in the first equation. Doing this, we obtain the following equivalent system:

\[
\begin{align*}
(c_{11}c_{22} - c_{12}c_{21})x_1 &= 0 \\
c_{22}x_2 &= 0
\end{align*}
\]

It is not difficult to see that this system has a nontrivial solution if and only if \(c_{11}c_{22} - c_{12}c_{21} = 0\). Hence, the determinant of the 2 by 2 (coefficient) matrix

\[
C = \begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}
\]

should be defined to be the number \(c_{11}c_{22} - c_{12}c_{21}\).

**Definition 3.1:** For a 1 by 1 or a 2 by 2 matrix \(C\), its \textit{determinant} is the number \(\det(C)\) defined as follows:

1. If \(C = [c_{11}]\), then \(\det(C) = c_{11}\).
2. If \(C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\), then \(\det(C) = c_{11}c_{22} - c_{12}c_{21}\).
Example 3.2: Consider the general homogeneous system of three equations in three variables:

\[
\begin{align*}
    c_{11}x_1 + c_{12}x_2 + c_{13}x_3 &= 0 \\
    c_{21}x_1 + c_{22}x_2 + c_{23}x_3 &= 0 \\
    c_{31}x_1 + c_{32}x_2 + c_{33}x_3 &= 0
\end{align*}
\] (H)

Assume, without loss of generality, that \(c_{33} \neq 0\). Following the method of substitution, let’s multiply each of the first two equations by \(c_{33}\), solve for \(c_{33}x_3\) in the last equation to obtain

\[c_{33}x_3 = -c_{31}x_1 - c_{32}x_2\]

and then substitute the expression \(-c_{31}x_1 - c_{32}x_2\) for \(c_{33}x_3\) in each of the first two equations to obtain the following system \(H'\):

\[
\begin{align*}
    (c_{11}c_{33} - c_{13}c_{31})x_1 &+ (c_{12}c_{33} - c_{13}c_{32})x_2 = 0 \\
    (c_{21}c_{33} - c_{23}c_{31})x_1 &+ (c_{22}c_{33} - c_{23}c_{32})x_2 = 0
\end{align*}
\] (H')

Note that the system \(H\) has a nontrivial solution if and only the system \(H'\) has a nontrivial solution.

Now then, \(H'\) is a homogeneous system with two equations and two variables. Hence, from what we’ve already found for the case \(n = 2\), \(H'\) has a nontrivial solution if and only if the determinant of its coefficient matrix is equal to 0. That is, \(H'\) has a nontrivial solution if and only if \(\det(C') = 0\), where

\[
C' = \begin{bmatrix}
    c_{11}c_{33} - c_{13}c_{31} & c_{12}c_{33} - c_{13}c_{32} \\
    c_{21}c_{33} - c_{23}c_{31} & c_{22}c_{33} - c_{23}c_{32}
\end{bmatrix}
\]

We leave it to the reader to show that \(\det(C') = 0\) if and only if

\[
c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33} - c_{13}c_{22}c_{31} = 0
\]

Thus, it makes sense to define the determinant of the 3 by 3 (coefficient) matrix

\[
C = \begin{bmatrix}
    c_{11} & c_{12} & c_{13} \\
    c_{21} & c_{22} & c_{23} \\
    c_{31} & c_{32} & c_{33}
\end{bmatrix}
\]

to be the number \(c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33} - c_{13}c_{22}c_{31}\).

Definition 3.2: The determinant of the 3 by 3 matrix

\[
C = \begin{bmatrix}
    c_{11} & c_{12} & c_{13} \\
    c_{21} & c_{22} & c_{23} \\
    c_{31} & c_{32} & c_{33}
\end{bmatrix}
\]

is the number \(\det(C)\) defined by

\[
\det(C) = c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33} - c_{13}c_{22}c_{31}
\]
Example 3.3: Find the determinant of each of the following 3 by 3 matrices:

(a) $C_1 = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 4 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

(b) $C_2 = \begin{bmatrix} 1 & -4 & 3 \\ -2 & 6 & -7 \\ 1 & 0 & 5 \end{bmatrix}$

Solution:

(a) For $C_1$, we find that

$$\det(C_1) = (1)(4)(1) + (-1)(3)(1) + (0)(3)(2) - (1)(3)(2) - (-1)(3)(1) - (0)(4)(1) = -2$$

Hence, $C_1$ is invertible. Or, equivalently, the homogeneous system $C_1X = Z$ has only the trivial solution.

(b) For $C_2$, we have that

$$\det(C_2) = (1)(6)(5) + (-4)(-7)(1) + (3)(-2)(0) - (1)(-7)(0) - (-4)((-2)(5) - (3)(6)(1) = 0$$

Thus, $C_2$ is not invertible, and the homogeneous system $C_2X = Z$ has nontrivial solutions. (What is the solution set of $C_2X = Z$?)

To see how we might generalize the definition of the determinant to cover matrices of size $n$ by $n$ for $n \geq 4$, let us look more closely at the case $n = 3$.

Note that the determinant of a 3 by 3 matrix $C$ is a sum of six terms, and each term is (plus or minus) a product of three entries from the matrix. Looking more closely at these three entries, note that no two of them come from the same row or column of the matrix. In other words, each term has the form

$$\pm C(1, j_1)C(2, j_2)C(3, j_3)$$

where $(j_1, j_2, j_3)$ is a permutation of $\{1, 2, 3\}$, the set of column positions for the matrix. In fact, the six different terms correspond to the six different permutations of $\{1, 2, 3\}$, namely:

$$(1, 2, 3) \quad (2, 3, 1) \quad (3, 1, 2) \quad (1, 3, 2) \quad (2, 1, 3) \quad (3, 2, 1)$$

(Recall that a permutation of the set $\{1, 2, \ldots, n\}$ is an arrangement of the numbers in this set into a sequence $(s_1, s_2, \ldots, s_n)$ such that each number occurs exactly once in the sequence. It is well-known that there are $n!$ ($n$-factorial) different permutations of $\{1, 2, \ldots, n\}$, where $n! = n(n-1) \cdots (2)(1)$. In particular, the number of permutations of $\{1, 2, 3\}$ is $3! = 3(2)(1) = 6$.)

Thus, we see that we could define the determinant of a 3 by 3 matrix $C$ as follows:

$$\det(C) = \sum \sigma(j_1, j_2, j_3)C(1, j_1)C(2, j_2)C(3, j_3)$$

where the sum is over all six permutations $(j_1, j_2, j_3)$ of $\{1, 2, 3\}$, and where $\sigma(j_1, j_2, j_3)$ is either +1 or −1 and is called the sign of the permutation $(j_1, j_2, j_3)$. Moreover, a similar definition works for 1 by 1 and for 2 by 2 matrices. For instance, note that the determinant of a 2 by 2 matrix $C$ can be defined as

$$\det(C) = \sum \sigma(j_1, j_2)C(1, j_1)C(2, j_2)$$

Here, the sum is over the 2 permutations of $\{1, 2\}$. 

To generalize then, we should expect the definition of the determinant of an \( n \) by \( n \) matrix \( C \) to have the following form:

\[
\det(C) = \sum \sigma(j_1, j_2, \ldots, j_n)C(1, j_1)C(2, j_2) \cdots C(n, j_n)
\]

where the sum is taken over all \( n! \) permutations \((j_1, j_2, \ldots, j_n)\) of \( \{1, 2, \ldots, n\} \). To complete the definition, we need to figure out how to determine the sign \( \sigma(j_1, j_2, \ldots, j_n) \) of a given permutation \((j_1, j_2, \ldots, j_n)\).

We define the following relation \( \sim \) on the set of permutations of \( \{1, 2, \ldots, n\} \): given permutations \( \alpha \) and \( \beta \),

\[\alpha \sim \beta \iff \beta \text{ can be obtained from } \alpha \text{ by interchanging (the positions of) two elements}\]

For instance, in the case \( n = 4 \), let \( \alpha = (2, 3, 1, 4) \) and \( \beta = (4, 3, 1, 2) \). Starting with \( \alpha \), if we interchange the elements 2 and 4, then we obtain \( \beta \). Hence, \( \alpha \sim \beta \). Also, \( \beta \sim \alpha \), since, starting with \( \beta \), if we interchange 2 and 4, we get \( \alpha \).

In fact, the relation \( \sim \) is easily seen to be symmetric; that is, for any two permutations \( \alpha \) and \( \beta \), if \( \alpha \sim \beta \), then \( \beta \sim \alpha \). Hence, we can consider the graph \( G_n \) of the relation \( \sim \); the nodes of this graph are the \( n! \) permutations of \( \{1, 2, \ldots, n\} \), and two nodes are joined with an edge if and only if the associated permutations are related under \( \sim \).

**Example 3.4:** Draw each of the graphs \( G_1 \), \( G_2 \), and \( G_3 \). Based on these graphs, is there a nice way to determine the sign of a given permutation of \( \{1, 2, \ldots, n\} \) for \( n \leq 3 \)?

**Solution:** We will go over this example in class. The graph \( G_3 \) is shown in Figure 3.1.
We are now prepared to define the determinant of an \( n \times n \) matrix.

**Definition 3.3:** For an \( n \times n \) matrix \( C \), the **determinant** of \( C \) is the number \( \det(C) \) defined as follows:

\[
\det(C) = \sum_{\sigma} \sigma(j_1, j_2, \ldots, j_n)C(1, j_1)C(2, j_2) \cdots C(n, j_n)
\]

where the sum is taken over all \( n! \) permutations of \( \{1, 2, \ldots, n\} \), and where the sign \( \sigma(j_1, j_2, \ldots, j_n) \) of a given permutation \( \alpha = (j_1, j_2, \ldots, j_n) \) is defined as follows:

\[
\sigma(\alpha) = \begin{cases} 
-1 & \text{if } \alpha \text{ is odd} \\
+1 & \text{if } \alpha \text{ is even}
\end{cases}
\]

Based on Example 3.4, a given permutation \( \alpha \) of \( \{1, 2, \ldots, n\} \) is odd or even based on whether its distance from the identity permutation \( (1, 2, \ldots, n) \) in \( G_n \) is odd or even. Alternatively, for those of you with some computer science background, think of sorting \( \alpha \) to obtain the identity permutation — if, in doing this, an odd number of interchanges are made, then \( \alpha \) is odd; if an even number of interchanges are made, then \( \alpha \) is even.

**Example 3.5:** Use Definition 3.3 to compute the determinant of the following matrix:

\[
C = \begin{bmatrix} 
0 & 2 & 0 & 0 \\
-1 & 0 & 4 & 0 \\
1 & 0 & 0 & 3 \\
6 & 0 & -5 & -2
\end{bmatrix}
\]

**Solution:** Each term in the determinant of \( C \) has the form

\[
\sigma(j_1, j_2, j_3, j_4)C(1, j_1)C(2, j_2)C(3, j_3)C(4, j_4)
\]

where \( C(1, j_1) \), \( C(2, j_2) \), \( C(3, j_3) \), and \( C(4, j_4) \) are entries chosen from rows 1, 2, 3, and 4, respectively, no two of which are in the same column. Note that, if any one of these entries is 0, then the term will be 0, and hence that term can be ignored in the computation of the determinant. Thus, we need consider only the nonzero terms, which means that we need to figure out how to choose nonzero entries \( C(1, j_1) \), \( C(2, j_2) \), \( C(3, j_3) \), and \( C(4, j_4) \) from rows 1, 2, 3, and 4, respectively, such that no two of these entries come from the same column of \( C \). This leads to the following:

\[
\det(C) = \sigma(2, 1, 4, 3)C(1, 2)C(2, 1)C(3, 4)C(4, 3) + \\
\sigma(2, 3, 1, 4)C(1, 2)C(2, 3)C(3, 1)C(4, 4) + \\
\sigma(2, 3, 4, 1)C(1, 2)C(2, 3)C(3, 4)C(4, 1)
\]

Note that the permutations \( (2, 1, 4, 3) \) and \( (2, 3, 4, 1) \) are even, whereas the permutation \( (2, 3, 4, 1) \) is odd. Thus,

\[
\det(C) = (2)(-1)(3)(-5) + (2)(4)(1)(-2) - (2)(4)(3)(6) = -130
\]

■
In the formula
\[ \det(C) = \sum \sigma(j_1, j_2, \ldots, j_n)C(1, j_1)C(2, j_2) \cdots C(n, j_n) \]
for computing the determinant of an \( n \times n \) matrix \( C \), the various products \( C(1, j_1)C(2, j_2) \cdots C(n, j_n) \) are called **elementary products**, and the various terms
\[ \sigma(j_1, j_2, \ldots, j_n)C(1, j_1)C(2, j_2) \cdots C(n, j_n) \]
are called **signed elementary products**. To compute the determinant of an \( n \times n \) matrix using the above formula, one must compute each of the \( n! \) signed elementary products. Since the value of \( n! \) increases rapidly as \( n \) increases, this approach may be infeasible for even moderate values of \( n \). For example, \( 10! = 3628800 \), and \( 20! \) exceeds \( 2.4 \times 10^{18} \).

Thus, we require a more efficient method for computing determinants. In the next section, we develop a method that computes the determinant of an \( n \times n \) matrix using \( O(n^3) \) operations.

An alternate notation for \( \det(C) \) is \( |C| \). For instance, the determinant of the 2 by 2 matrix
\[
\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}
\]
could be denoted by
\[ \det(C) \quad \text{or} \quad \det\left(\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}\right) \quad \text{or} \quad |C| \quad \text{or} \quad \begin{vmatrix}
2 & 3 \\
5 & 7
\end{vmatrix} \]

**Exercise Set 3.1**

1. Use Definition 3.1 to compute the determinant of each of the following matrices.
   (a) \( C_0 = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \)
   (b) \( C_1 = \begin{bmatrix} 4 & 3 \\ 9 & 8 \end{bmatrix} \)
   (c) \( C_2 = E(1; 2, 1)C_1 \)
   (d) \( C_3 = E(2; 1, 2)C_1 \)
   (e) \( C_4 = E(3; -2, 1, 2)C_1 \)
   (f) \( C_5 = E(3; -2, 1, 2)C_0 \)

2. For an \( n \times n \) matrix \( C \), prove:
   (a) If \( C \) has a row of zeros, then \( \det(C) = 0 \).
   (b) If \( C \) has a column of zeros, then \( \det(C) = 0 \).

3. Use Definition 3.2 to compute the determinant of each of the following matrices.
   (a) \( C_0 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \)
   (b) \( C_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \)
   (c) \( C_2 = E(1; -3, 1)C_1 \)
   (d) \( C_3 = E(2; 2, 3)C_1 \)
   (e) \( C_4 = E(3; -2, 1, 2)C_1 \)
   (f) \( C_5 = E(3; 1, 1, 2)C_0 \)

4. Verify, for any numbers \( a, b, \) and \( c \), that
\[
\begin{vmatrix}
1 & 1 & 1 \\
-1 & b & c \\
-1 & b^2 & c^2
\end{vmatrix}
= (a - b)(b - c)(c - a)
\]

5. Use Definition 3.3 to compute the determinant of each of the following matrices (where \( a, b, c, \ldots \) represent arbitrary numbers).
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6. For a 3 by 3 matrix $C$, verify that $\det(C) = \det(C^T)$. It turns out that this result holds for square matrices of any size.

7. Let $C$ be an $n$ by $n$ matrix. Prove: If every entry of $C$ is an integer, then $\det(C)$ is an integer.

3.2 COMPUTING DETERMINANTS USING ELEMENTARY OPERATIONS

We noted in the last section that using Definition 3.3 to compute the determinant of an $n$ by $n$ matrix requires $O(n!)$ steps, in general, since each of the $n!$ signed elementary products must be determined. On the other hand, transforming an $n$ by $n$ matrix $C$ to (reduced) echelon form requires only $O(n^3)$ arithmetic operations, in general, since performing an elementary row operation on $C$ requires $O(n)$ arithmetic operations, and the number of elementary row operations required is $O(n^2)$ (since $C$ has $n^2$ entries).

In this section, we give a method for computing the determinant of an $n$ by $n$ matrix $C$ in $O(n^3)$ steps. The idea is to perform a sequence of elementary row operations on $C$ to obtain a matrix $U$ that is upper triangular. In particular, any matrix in echelon form is upper triangular, and so finding $U$ requires no more than $O(n^3)$ arithmetic operations. It will be seen that computing the determinant of $U$ is easy, since each of its elementary products, with at most one exception, involves a factor of zero. Moreover, performing an elementary row operation on a matrix changes the value of the determinant in a predictable way, and hence we can obtain the value of $\det(C)$ from the value of $\det(U)$.

Now then, an $n$ by $n$ matrix $U$ is said to be upper triangular provided each of its entries below the main diagonal is 0; that is, $U$ is upper triangular if and only if $U(i,j) = 0$ whenever $i > j$. Similarly, an $n$ by $n$ matrix $L$ is said to be lower triangular provided each of its entries above the main diagonal is zero. A matrix that is either lower triangular or upper triangular is called triangular.

Example 3.6: The following matrices $U_2$, $U_3$, and $U_4$ are upper triangular for any numbers $a$, $b$, $c$, ...:

\[
U_2 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad U_3 = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \quad U_4 = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}
\]
(a) Use Definition 3.3 to compute \( \det(U_2) \), \( \det(U_3) \), and \( \det(U_4) \).

(b) Give similar examples of lower triangular matrices \( L_2 \), \( L_3 \), and \( L_4 \), and compute their determinants.

Solution: We do part (a) and leave part (b) for the student.

Applying Definition 3.3, we find that

\[
\det(U_2) = ac - b(0) = ac
\]

and

\[
\det(U_3) = adf + be(0) + c(0)(0) - cd(0) - ae(0) - b(0)(0) = adf
\]

For \( U_4 \), we note that the only elementary product which is nonzero is \( aehj \), and it has a positive sign. Thus, in all three cases, the determinant of the given upper-triangular matrix is the product of the entries on the main diagonal.

Taking our lead from the preceding example, we obtain the following result.

**Theorem 3.1:** For an \( n \) by \( n \) upper triangular matrix \( U \), \( \det(U) \) is equal to the product of the entries on the main diagonal; that is,

\[
\det(U) = U(1,1)U(2,2)\cdots U(n,n)
\]

**Proof:** It suffices to verify that, in the computation of \( \det(U) \), every elementary product is zero with the possible exception of \( U(1,1)U(2,2)\cdots U(n,n) \). Furthermore, the elementary product \( U(1,1)U(2,2)\cdots U(n,n) \) corresponds to the identity permutation \((1,2,\ldots,n)\) of the column indices of \( U \), which is an even permutation, and hence this elementary product has a positive sign.

Consider an arbitrary elementary product of \( U \), say \( U(i_1,j_1)U(i_2,j_2)\cdots U(i_n,j_n) \). Since this product contains exactly one factor from each row of \( U \), we may rewrite it in the form

\[
U(i_1,1)U(i_2,2)\cdots U(i_n,n)
\]

where \((i_1,i_2,\ldots,i_n)\) is a permutation of the row indices. Now then, since \( U \) is upper triangular, \( U(r,1) = 0 \) for \( r > 1 \). Hence, if we are trying to make the above elementary product nonzero, we may assume that \( i_1 = 1 \). Next, consider, the factor \( U(i_2,2) \). Again, since \( U \) is upper triangular, \( U(r,2) = 0 \) for \( r > 2 \). Thus, in order to have \( U(i_2,2) \) nonzero, we are forced to let \( i_2 = 2 \) (since \( i_2 \neq i_1 = 1 \)). Continuing in this fashion, we see that the only way the above elementary product can be nonzero is if \( i_k = k, 1 \leq k \leq n \). This completes the proof.

We mention that a similar result holds for lower triangular matrices, namely, if \( L \) is an \( n \) by \( n \) lower triangular matrix, then \( \det(L) = L(1,1)L(2,2)\cdots L(n,n) \).

Recall that, given a matrix, there are three types of elementary row operations that can be performed on it:

- **Type 1:** multiply some row of the matrix by some nonzero constant \( c \);  
- **Type 2:** interchange some two rows of the matrix;  
- **Type 3:** add some multiple of one row to some other row.
An elementary matrix is a square matrix that is obtained by performing a single elementary row operation on the identity matrix $I$ of the same size. We have adopted the following notation for elementary matrices:

- $E(1; c, i) = \text{the matrix obtained from } I \text{ by multiplying the } i \text{th row by } c$
- $E(2; i_1, i_2) = \text{the matrix obtained from } I \text{ by interchanging rows } i_1 \text{ and } i_2$
- $E(3; c, i_1, i_2) = \text{the matrix obtained from } I \text{ by adding } c \text{ times row } i_1 \text{ to row } i_2$

The determinant of an elementary matrix is easily computed; in particular, note that any type I or type III elementary matrix is triangular.

**Theorem 3.2:** Let $I$ denote the $n \times n$ identity matrix, let $c$ be a nonzero constant, and let $i, i_1,$ and $i_2$ be integers between 1 and $n$. Then:

1. $\det(I) = 1$
2. $\det(E(1; c, i)) = c$
3. $\det(E(2; i_1, i_2)) = -1$
4. $\det(E(3; c, i_1, i_2)) = 1$

**Proof:** The results in parts 1, 2, and 4 follow immediately from Theorem 3.1 (or the corresponding result for lower-triangular matrices).

For part 3, note that exactly one elementary product in the matrix $E(2; i_1, i_2)$ is nonzero; in fact, the value of this product is 1. The permutation associated with this elementary product is odd, since it results from the identity permutation $(1, 2, \ldots, n)$ by swapping the values $i_1$ and $i_2$. In other words, the matrix $E(2; i_1, i_2)$ has exactly one nonzero entry in each row, these nonzero entries are all 1s, and all these 1s occur on the main diagonal with precisely two exceptions: the 1 in row $i_1$ occurs in column $i_2$ and the 1 in row $i_2$ occurs in column $i_1$. It follows that the determinant of $E(2; i_1, i_2)$ is $-1$.

Let $C$ be an $n \times n$ matrix, and let $C_1$ be the matrix obtained by performing the elementary operation $E_1$ on $C$; that is, $C_1 = E_1C$, where $E_1$ is the corresponding elementary matrix. How is $\det(C_1)$ related to $\det(C)$? To help us answer this question, let’s consider an example. (You may also wish to review the results of Exercises 1 and 3 of Exercise Set 3.1.)

**Example 3.7:** Consider the following matrix:

\[
C = C_0 = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 3 \\ -2 & -3 & 3 \end{bmatrix}
\]

We perform a sequence of elementary row operations on the matrix $C$, at least one of each type, transforming $C$ into a matrix $U$ that is upper triangular.
First, we compute the determinant of $C$:
\[
\det(C) = 2(4)(3) + 4(3)(-2) + 2(0)(-3) - 2(4)(-2) - 2(3)(-3) - 4(0)(3) = 34
\]

Let $E_1 = E(3; 1, 1, 3)$ and let $C_1 = E_1 C$. Then

\[
C_1 = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 3 \\ 0 & 1 & 5 \end{bmatrix}
\]

and
\[
\det(C_1) = 2(4)(5) + 4(3)(0) + 2(0)(1) - 2(4)(0) - 2(3)(1) - 4(0)(5) = 34
\]

Thus, we see that $\det(C_1) = \det(C) = 1 \cdot \det(C) = \det(E_1) \cdot \det(C)$. 

Next, let $E_2 = E(2; 2, 3)$ and let $C_2 = E_2 C_1$. Then

\[
C_2 = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 4 & 3 \end{bmatrix}
\]

and
\[
\det(C_2) = 2(1)(3) + 4(5)(0) + 2(0)(4) - 2(1)(0) - 2(5)(4) - 4(0)(3) = -34
\]

Note that $\det(C_2) = -\det(C_1) = \det(E_2) \cdot \det(C_1)$. (In fact, to further reinforce the result of Theorem 3.2, part 3, note that $\det(C_2)$ has exactly the same elementary products as $\det(C_1)$, but each one has the opposite sign in $\det(C_2)$."

Next, to see the effect of a type 1 operation, let $E_3 = E(1; 1/2, 1)$ and let $C_3 = E_3 C_2$. Then

\[
C_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & 4 & 3 \end{bmatrix}
\]

and
\[
\det(C_3) = 1(1)(3) + 2(5)(0) + 2(0)(4) - 2(1)(0) - 1(5)(4) - 2(0)(3) = -17
\]

Note that $\det(C_3) = (1/2) \det(C_2) = \det(E_3) \cdot \det(C_2)$. (In fact, notice that each elementary product in $\det(C_3)$ is $1/2$ the corresponding elementary product in $\det(C_2)$.)

Finally, let $E_4 = E(3; -4, 2, 3)$ and let $C_4 = E_4 C_3$. Then

\[
C_4 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & -17 \end{bmatrix}
\]

Since $C_4$ is triangular, we have that $\det(C_4) = 1(1)(-17) = -17$. Note that $\det(C_4) = \det(C_3) = \det(E_4) \cdot \det(C_3)$.

Thus, for each $k$, $0 \leq k \leq 3$, we have shown that $\det(C_{k+1}) = \det(E_{k+1}) \det(C_k)$. Knowing this, and knowing how to compute determinants of elementary matrices, we could have computed $\det(C)$ as follows:

\[
C_4 = E_4 E_3 E_2 E_1 C \rightarrow \det(C_4) = \det(E_4) \det(E_3) \det(E_2) \det(E_1) \det(C) \\
\rightarrow -17 = 1(1/2)(-1)(1) \det(C)
\]

Hence, $\det(C) = 34$. 

\[\blacksquare\]
**Theorem 3.3:** For any $n$ by $n$ matrix $C$ and any $n$ by $n$ elementary matrix $E$,

$$\det(EC) = \det(E) \det(C)$$

In particular:

1. If $C_1 = E(1;c,i)C$, then $\det(C_1) = c \cdot \det(C)$.
2. If $C_2 = E(2;i_1,i_2)C$, then $\det(C_2) = -\det(C)$.
3. If $C_3 = E(3;c,i_1,i_2)C$, then $\det(C_3) = \det(C)$.

**Proof:** We prove part 1, comment on part 2, and leave part 3 to Exercise 9.

For part 2, the proof hinges on observing that the matrices $C$ and $C_2$ have exactly the same elementary products, but precisely the opposite signed elementary products; that is, a given elementary product has a positive sign in $C$ if and only if it has a negative sign in $C_2$. Therefore, $\det(C_2) = -\det(C)$.

For part 1, let’s first handle the case when $i = 1$. In this case,

$$\det(C_1) = \sum \sigma(j_1,j_2,\ldots,j_n)C_1(1,j_1)C_1(2,j_2) \cdots C_1(n,j_n)$$

$$= \sum \sigma(j_1,j_2,\ldots,j_n)c \cdot C(1,j_1)C(2,j_2) \cdots C(n,j_n)$$

$$= c \sum \sigma(j_1,j_2,\ldots,j_n)C(1,j_1)C(2,j_2) \cdots C(n,j_n)$$

$$= c \cdot \det(C)$$

Now, to handle the case when $i > 1$, let $E_1 = E(1;c,1)$ and $E_2 = E(2;1,i)$. Then $C_1 = E_2E_1E_2C$, and it follows from part 2 and the special case of part 1 that we have proven above that

$$\det(C_1) = \det(E_2E_1E_2C)$$

$$= \det(E_2) \cdot \det(E_1) \cdot \det(E_2) \cdot \det(C)$$

$$= (-1)(c)(-1) \cdot \det(C)$$

$$= c \cdot \det(C)$$

Theorems 3.1, 3.2, and 3.3 provide the desired efficient method for computing determinants. Given an $n$ by $n$ matrix $C$, if $C$ is triangular, then its determinant is easily computed as the product of the main-diagonal entries. If $C$ is not triangular, then we perform a sequence of elementary row operations on $C$ to transform it to a matrix $U$ that is upper triangular; that is, we find elementary matrices $E_1, E_2, \ldots, E_k$ such that $U = E_k \cdots E_2E_1C$ and $U$ is upper triangular. Then, by Theorem 3.3,

$$\det(U) = \det(E_k) \cdots \det(E_2) \det(E_1) \det(C)$$

Since $\det(U)$ and the determinants of the elementary matrices are easily computed, we can then find $\det(C)$. 
Example 3.8: Compute the determinant of the following matrix:

\[
C = \begin{bmatrix}
1 & 2 & -5 & 1 \\
-4 & -3 & 5 & -4 \\
2 & 8 & -6 & 1 \\
-1 & -2 & 7 & -2
\end{bmatrix}
\]

Solution: Define the following elementary matrices:

\[
E_1 = E(3; 4, 1, 2) \\
E_2 = E(3; -2, 1, 3) \\
E_4 = E(1; 1/5, 2) \\
E_5 = E(3; -4, 2, 3) \\
E_7 = E(3; -8, 3, 4)
\]

and let \( C_7 = E_7E_6E_5E_4E_3E_2E_1C \). The student should verify that

\[
C_7 = \begin{bmatrix}
1 & 2 & -5 & 1 \\
0 & 1 & -3 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & 0 & 7
\end{bmatrix}
\]

Then

\[
14 = \det(C_7) = \prod_{i=1}^{7} \det(E_i) \cdot \det(C) = (-1/5) \cdot \det(C)
\]

It follows that \( \det(C) = -70 \).

Exercise Set 3.2

1. Use the method of this section to compute each of the following determinants.

(a) \[
\begin{vmatrix}
4 & 3 \\
9 & 8
\end{vmatrix}
\]

(b) \[
\begin{vmatrix}
1 & -1 & 0 \\
3 & 4 & 3 \\
1 & 2 & 1
\end{vmatrix}
\]

(c) \[
\begin{vmatrix}
1 & 1 & 1 & -2 \\
2 & 2 & 3 & -6 \\
1 & 1 & 0 & 0 \\
3 & 3 & 2 & -4
\end{vmatrix}
\]

2. For an \( n \) by \( n \) matrix \( C \), prove that if one row is a multiple of another row (in particular, if two rows of \( C \) are the same), then \( \det(C) = 0 \).

3. Use the method of this section to compute each of the following determinants.

(a) \[
\begin{vmatrix}
1 & 0 & -2 & 1 \\
0 & 1 & 3 & -1 \\
3 & 2 & -1 & 1 \\
1 & -2 & 2 & 3
\end{vmatrix}
\]

(b) \[
\begin{vmatrix}
2 & -1 & 4 & 0 & -1 \\
-1 & 2 & -2 & -3 & -1 \\
1 & -2 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & -1 \\
3 & -6 & 3 & 0 & 4
\end{vmatrix}
\]

4. Use the method of this section to redo Exercise 4 of Exercise Set 3.1; that is, show that

\[
\begin{vmatrix}
1 & 1 & 1 \\
a & b & c \\
a^2 & b^2 & c^2
\end{vmatrix} = (a - b)(b - c)(c - a)
\]
5. Compute the determinant of each of the following matrices:

\[
\begin{bmatrix}
4 & 2 & 0 & -2 \\
2 & 2 & -3 & -1 \\
0 & -3 & 10 & 0 \\
-2 & -1 & 0 & 2
\end{bmatrix}
\]

(a) \[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
-1 & -2 & 0 & 1 \\
2 & -1 & 1 & 0 \\
-2 & 1 & 1 & 0
\end{bmatrix}
\]

6. For any \(n\) by \(n\) matrix \(C\) and any number \(k\), prove: \(\det(kC) = k^n \det(C)\).

7. Given that

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix} = 5
\]

find:

\[
\begin{vmatrix}
g & h & i \\
d & e & f \\
a & b & c
\end{vmatrix}
\]

(a) \[
\begin{vmatrix}
-a & -b & -c \\
2d & 2e & 2f \\
-3g & -3h & -3i
\end{vmatrix}
\]

(b) \[
\begin{vmatrix}
d + g & e + h & f + i \\
d & e & f \\
a & b & c
\end{vmatrix}
\]

8. Recall (from Chapter 1, Problem 16) that a square matrix \(C\) is called skew-symmetric if \(C^T = -C\). Prove:

(a) If \(C\) is an \(n\) by \(n\) skew-symmetric matrix, then \(\det(C) = (-1)^n \det(C)\).

(b) If \(C\) is an \(n\) by \(n\) skew-symmetric matrix and \(n\) is odd, then \(\det(C) = 0\).

9. Prove Theorem 3.3, part 3. Hint: First handle the case when \(i_1 = 1\) and \(i_2 = 2\).

3.3 ADDITIONAL PROPERTIES OF THE DETERMINANT FUNCTION

Recall that our goal in this chapter is to define the determinant of a square matrix \(C\) in such a way that

\[C\text{ is invertible } \iff \det(C) \neq 0\]

So far we have defined the determinant function, but we still need to show that it has the above desired property. We do that presently, but first we prove the following useful lemma.

Lemma 3.4: For \(n\) by \(n\) matrices \(C_1\) and \(C_2\), if \(C_1\) and \(C_2\) are row equivalent, then \(\det(C_1)\) and \(\det(C_2)\) are either both zero or are both nonzero.

Proof: Assume \(C_1\) and \(C_2\) are row equivalent. Then there exist elementary matrices \(E_1, E_2, \ldots, E_k\) such that

\[C_2 = E_k \cdots E_2 E_1 C_1\]
It follows from Theorem 3.3 that
\[ \det(C_2) = \det(E_k) \cdots \det(E_2) \det(E_1) \det(C_1) \]
Now then, by Theorem 3.2, the determinant of any elementary matrix is nonzero, and hence the product \( \det(E_k) \cdots \det(E_2) \det(E_1) \) is nonzero. It follows that \( \det(C_1) = 0 \) if and only \( \det(C_2) = 0 \).

**Theorem 3.5:** For any \( n \times n \) matrix \( C \), \( C \) is invertible if and only if \( \det(C) \neq 0 \).

**Proof:** Let \( C^* \) be the matrix in reduced echelon form to which \( C \) is row equivalent.

For necessity, assume \( C \) is invertible. Then, by Theorem 2.5, \( C^* = I \), and hence \( \det(C^*) = 1 \neq 0 \). Thus, by Lemma 3.4, \( \det(C) \neq 0 \).

For sufficiency, we prove the contrapositive:

If \( C \) is not invertible, then \( \det(C) = 0 \).

Assume \( C \) is not invertible. Then \( C^* \) contains a row of zeros, and it follows that \( \det(C^*) = 0 \) (see Exercise 2 in Exercise Set 3.1). Thus, by Lemma 3.4, \( \det(C) = 0 \).

Next we show that the determinant function distributes over a matrix product.

**Theorem 3.6:** If \( A \) and \( B \) are square matrices with the same size, then
\[ \det(AB) = \det(A) \det(B) \]

**Proof:** Let \( A \) and \( B \) be \( n \times n \) matrices. We consider two cases, depending on whether or not \( A \) is invertible.

**Case 1:** \( A \) is invertible. Then, by Theorem 2.5, we have that \( A \) is a product of elementary matrices, say,
\[ A = E_1 E_2 \cdots E_k \]
Hence, by Theorem 3.3,
\[ \det(AB) = \det(E_1 E_2 \cdots E_k B) = \det(E_1) \det(E_2) \cdots \det(E_k) \det(B) = \det(E_1 E_2 \cdots E_k) \det(B) = \det(A) \det(B) \]

**Case 2:** \( A \) is not invertible. Then, by the result of Exercise 10 of Exercise Set 2.4, the product \( AB \) is also not invertible. Thus, by Theorem 3.5,
\[ \det(AB) = 0 = 0 \cdot \det(B) = \det(A) \det(B) \]
Corollary 3.7: For an \( n \) by \( n \) matrix \( C \), if \( C \) is invertible, then
\[
\det(C^{-1}) = \frac{1}{\det(C)}
\]

**Proof:** Assume \( C \) is invertible. Then \( C^{-1} \) exists, and \( CC^{-1} = I \). Hence, by Theorem 3.6,
\[
1 = \det(I) = \det(CC^{-1}) = \det(C) \det(C^{-1})
\]
This indicates that the numbers \( \det(C) \) and \( \det(C^{-1}) \) are reciprocals of each other.

We conclude this section with the following summary result, which extends Theorem 2.5 and follows immediately from Theorem 3.5.

**Theorem 3.8 (First Fundamental Theorem of Linear Algebra):** For a given \( n \) by \( n \) matrix \( C \), the following statements are equivalent:

1. \( C \) is invertible.
2. The homogeneous system \( CX = Z \) has only the trivial solution.
3. \( C \) is row equivalent to the \( n \) by \( n \) identity matrix \( I \).
4. There exists a sequence \( E_1, E_2, \ldots, E_k \) of elementary matrices such that
\[
E_k \cdots E_2 E_1 C = I
\]
5. \( \det(C) \neq 0 \).

**Exercise Set 3.3**

1. For an \( n \) by \( n \) matrix \( C \), prove: If \( C \) is invertible and every entry of \( C \) and of \( C^{-1} \) is an integer, then
\[
\det(C) = \pm 1
\]
2. For an \( n \) by \( n \) matrix \( C \), prove: If \( C \) is orthogonal, then \( \det(C) = \pm 1 \).
3. Let \( C \) be an \( n \) by \( n \) matrix each of whose row sums is zero; that is, for any \( i, 1 \leq i \leq n \),
\[
\sum_{j=1}^{n} C(i, j) = 0
\]
Find \( \det(C) \). (Hint: Does the homogeneous system \( CX = Z \) have a nontrivial solution?)
4. For an \( n \) by \( n \) matrix \( C \), prove: \( C \) is invertible if and only if \( C^T C \) is invertible.
5. For \( n \) by \( n \) matrices \( A \) and \( B \), prove: If \( A \) is invertible, then \( \det(B) = \det(A^{-1}BA) \).
6. For a square matrix \( C \) and a positive integer \( n \), prove (by induction on \( n \)) that
\[
\det(C^n) = (\det(C))^n
\]
3.4 COFACTOR EXPANSION AND CRAMER’S RULE

In this section, we develop an alternate method for computing the determinant of an \( n \times n \) matrix. Although not an efficient method in general, it does work well for certain matrices, and it also has theoretical value. Moreover, it leads to a formula for the inverse of an invertible matrix, and to a method for solving a system of \( n \) linear equations in \( n \) unknowns, provided the system is consistent and has a unique solution.

Let \( C \) be a 3 by 3 matrix. Recall that \( \det(C) \) is given by
\[
\det(C) = c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33} - c_{13}c_{22}c_{31}
\]

The last expression on the right-hand side above is called the cofactor expansion of \( \det(C) \) about the first row of \( C \).

**Definition 3.4:** Given an \( n \times n \) matrix \( C \):

1. The **minor** \( m(i, j) \) of the entry \( C(i, j) \) is the determinant of the \( n - 1 \times n - 1 \) matrix obtained by deleting the \( i \)th row and \( j \)th column of \( C \).
2. The **cofactor** \( c(i, j) \) of the entry \( C(i, j) \) is defined by
   
   \[
   c(i, j) = (-1)^{i+j} m(i, j)
   \]

   and the **cofactor matrix** of \( C \) is the \( n \times n \) matrix \( \text{cof}(C) \) whose entry in row \( i \), column \( j \) is \( c(i, j) \).
3. The **adjoint** of \( C \) is the \( n \times n \) matrix \( \text{adj}(C) \) defined by
   
   \[
   \text{adj}(C) = (\text{cof}(C))^T
   \]

With this definition, we may express the determinant of a 3 by 3 matrix \( C \) as
\[
\det(C) = C(1, 1)c(1, 1) + C(1, 2)c(1, 2) + C(1, 3)c(1, 3)
\]
namely, to compute \( \det(C) \) using a cofactor expansion about row 1, we multiple each entry in that row by its corresponding cofactor, and then sum these products. Our next result generalizes this idea.

**Theorem 3.9:** Given an \( n \times n \) matrix \( C \), its determinant may be computed using a cofactor expansion. In particular:

1. The **cofactor expansion** for \( \det(C) \) **about the \( i \)th row** is given by
   
   \[
   \det(C) = \sum_{j=1}^{n} C(i, j)c(i, j)
   \]
2. The \textit{cofactor expansion} for \( \det(C) \) \textit{about the \( j \)th column} is given by

\[
\det(C) = \sum_{i=1}^{n} C(i, j)c(i, j)
\]

\[\blacksquare\]

\textbf{Example 3.9:} Consider again the matrix \( C \) of Example 3.7:

\[
C = \begin{bmatrix}
2 & 4 & 2 \\
0 & 4 & 3 \\
-2 & -3 & 3
\end{bmatrix}
\]

Compute \( \det(C) \) using:

(a) cofactor expansion about row 1;

(b) cofactor expansion about column 1.

\textbf{Solution:}

(a)

\[
\det(C) = C(1, 1)c(1, 1) + C(1, 2)c(1, 2) + C(1, 3)c(1, 3)
\]

\[
\begin{align*}
&= 2 \begin{vmatrix}
4 & 3 \\
-3 & 3
\end{vmatrix} - 0 \begin{vmatrix}
0 & 3 \\
-2 & 3
\end{vmatrix} + 2 \begin{vmatrix}
0 & 4 \\
-2 & -3
\end{vmatrix} \\
&= 2(21) - 0(6) + 2(8) = 34
\end{align*}
\]

(b)

\[
\det(C) = C(1, 1)c(1, 1) + C(2, 1)c(2, 1) + C(3, 1)c(3, 1)
\]

\[
\begin{align*}
&= 2 \begin{vmatrix}
4 & 3 \\
-3 & 3
\end{vmatrix} - 0 \begin{vmatrix}
4 & 2 \\
-3 & 3
\end{vmatrix} + (-2) \begin{vmatrix}
4 & 2 \\
4 & 3
\end{vmatrix} \\
&= 2(21) - 0(18) + (-2)(4) = 34
\end{align*}
\]

\[\blacksquare\]

\textbf{Example 3.10:} Consider again the matrix \( C \) of Example 3.8:

\[
C = \begin{bmatrix}
1 & 2 & -5 & 1 \\
-4 & -3 & 5 & -4 \\
2 & 8 & -6 & 1 \\
-1 & -2 & 7 & -2
\end{bmatrix}
\]

When using a cofactor expansion to compute a determinant, it is often most efficient to perform the expansion about a row (or column) that contains the maximum number of zero entries, since the term \( C(i, j)c(i, j) \) in a cofactor expansion will be zero if the entry \( C(i, j) = 0 \). Here, our matrix \( C \) has no zero entries. However, let \( D \) be the matrix obtained from \( C \) by adding 4 times row 1 to row 2. Then

\[
D = \begin{bmatrix}
1 & 2 & -5 & 1 \\
0 & 5 & -15 & 0 \\
2 & 8 & -6 & 1 \\
-1 & -2 & 7 & -2
\end{bmatrix}
\]
Furthermore, \( \det(C) = \det(D) \). Find \( \det(D) \) using cofactor expansion about the second row.

**Solution:**

\[
\det(D) = \prod_{j=1}^{4} D(2, j) \cdot d(2, j)
\]

\[
= D(2, 2) \cdot d(2, 2) + D(2, 3) \cdot d(2, 3) \quad \text{\( \text{since} \ D(2, 1) = 0 = D(2, 4) \)}
\]

\[
= 5 \begin{vmatrix} 1 & -5 & 1 \\ 2 & -6 & 1 \\ -1 & 7 & -2 \end{vmatrix} - (-15) \begin{vmatrix} 1 & 2 & 1 \\ -1 & -2 & -2 \end{vmatrix}
\]

\[
= 5 \begin{vmatrix} 1 & -5 & 1 \\ 2 & -6 & 1 \\ 0 & 2 & -1 \end{vmatrix} + 15 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} \quad \text{(adding row 1 to row 3)}
\]

\[
= 5 \left( (-2) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & -5 \\ 2 & -6 \end{vmatrix} \right) + 15 \left( (-1) \begin{vmatrix} 1 & 2 \\ 2 & 8 \end{vmatrix} \right) \quad \text{(cofactor expansion, row 3)}
\]

\[
= 5((-2)(-1) + (-1)(4)) + 15((-1)(4)) = 5(-2) + 15(4) = 70
\]

In a cofactor expansion of \( \det(C) \) about a given row \( i_1 \), we compute \( \det(C) \) by multiplying each entry in row \( i_1 \) by its cofactor and then summing these products. It turns out that if we multiply each entry in row \( i_1 \) by the cofactor of the corresponding entry in a different row \( i_2 \), and then sum these products, the sum will be 0. (There is a corresponding result for columns.)

For example, consider the 3 by 3 matrix \( C \) from Example 3.9:

\[
C = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 3 \\ -2 & -3 & 3 \end{bmatrix}
\]

Let’s compute the value of the expression \( C(1, 1)c(3, 1) + C(1, 2)c(3, 2) + C(1, 3)c(3, 3) \), where we have used the entries from the first row and the cofactors of the entries from the third row:

\[
C(1, 1)c(3, 1) + C(1, 2)c(3, 2) + C(1, 3)c(3, 3) = 2 \begin{vmatrix} 4 & 2 \\ 4 & 3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} 2 & 4 \\ 0 & 4 \end{vmatrix}
\]

\[
= 2(4) - 4(6) + 2(8) = 0
\]

**Lemma 3.10:** For an \( n \) by \( n \) matrix \( C \) and for distinct integers \( i_1 \) and \( i_2 \) between 1 and \( n \),

\[
\sum_{j=1}^{n} C(i_1, j)c(i_2, j) = 0
\]

**Proof:** We give the proof in the case when \( i_1 = 1 \) and \( i_2 = 2 \); this will serve to illustrate how the result could be proved in general.
Let \( C' \) be the matrix obtained from \( C \) by replacing the second row by a copy of the first row. Then \( C' \) has two rows the same; hence, by the result of Exercise 2 of Exercise Set 3.2, \( \det(C') = 0 \). Thus,

\[
0 = \det(C') = C'(2, 1)c'(2, 1) + C'(2, 2)c'(2, 2) + \cdots + C'(2, n)c'(2, n)
\]

as was to be shown.

Theorem 3.11: Let \( C \) be an \( n \times n \) matrix and let \( I \) denote the \( n \times n \) identity matrix. Then

\[
C(\text{adj}(C)) = \det(C) \ I
\]

In words, the product of a matrix \( C \) and its adjoint is a diagonal matrix, with each entry on the diagonal equal to the determinant of \( C \).

Proof: For the matrix on the left-hand side, we compute a typical entry:

\[
[C(\text{adj}(C))(i, j) = C(i, \ast) \cdot \text{adj}(C)(\ast, j)
\]

\[
= C(i, \ast) \cdot \text{cof}(C)(j, \ast)
\]

\[
= \begin{cases} 
0 & \text{if } i \neq j \quad \text{(by Lemma 3.10)} \\
\det(C) & \text{if } i = j \quad \text{(by Theorem 3.9)}
\end{cases}
\]

\[
= \det(C) I
\]

The following corollary is an immediate consequence of Theorem 3.11.

Corollary 3.12: Let \( C \) be an \( n \times n \) matrix. If \( C \) is invertible, then

\[
C^{-1} = \frac{1}{\det(C)} \text{adj}(C)
\]

An important remark: To compute the inverse of a given matrix \( C \) using Corollary 3.12, it is not necessary to explicitly compute the determinant of \( C \). It is only necessary to compute the adjoint of \( C \), and then to check the product \( C(\text{adj}(C)) \) to be sure it has the form \( dI \) for some number \( d \). If it does, then \( d = \det(C) \), and if \( d \neq 0 \), then

\[
C^{-1} = \frac{1}{d} \text{adj}(C)
\]

Example 3.11: Use Corollary 3.12 to find the inverse of the matrix \( C \) of:
3.4 Cofactor Expansion and Cramer’s Rule

(a) Example 3.9

Solution: We do part (a) and leave part (b) to the student.

We first compute \( \text{adj}(C) \):

\[ \text{adj}(C) = \begin{vmatrix} c(1,1) & c(2,1) & c(3,1) \\ c(1,2) & c(2,2) & c(3,2) \\ c(1,3) & c(2,3) & c(3,3) \end{vmatrix} = \begin{bmatrix} 21 & -18 & 4 \\ -6 & 10 & -6 \\ 8 & -2 & 8 \end{bmatrix} \]

We then compute the product \( C(\text{adj}(C)) \):

\[ C(\text{adj}(C)) = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 3 \\ -2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 21 & -18 & 4 \\ -6 & 10 & -6 \\ 8 & -2 & 8 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix} \]

It follow that \( \det(C) = 34 \) and that

\[ C^{-1} = \frac{1}{34} \begin{bmatrix} 21 & -18 & 4 \\ -6 & 10 & -6 \\ 8 & -2 & 8 \end{bmatrix} \]

In general, the method given by Corollary 3.12 is not an efficient method for computing the inverse of an invertible matrix. Rather, the method given in Section 2.4 is the method of choice for general matrices. However, the formula for the inverse of an invertible matrix given by Corollary 3.12 does have some nice applications; see, for example, Exercise 2.

Our next result provides a useful formula, in terms of determinants, for solving a consistent system of \( n \) linear equations in \( n \) unknowns. It is named for the eighteenth century Swiss mathematician Gabriel Cramer (and not, unfortunately, for the character from *Seinfeld*).

**Theorem 3.13 (Cramer’s Rule):** Consider a system \( CX = B \) of \( n \) linear equations in \( n \) unknowns. For \( 1 \leq j \leq n \), let \( C_j \) be the matrix obtained by replacing the \( j \)th column of the coefficient matrix \( C \) by the vector \( B \). If \( \det(C) \neq 0 \), then the system has the unique solution \( (x_1, x_2, \ldots, x_n) \), where

\[ x_j = \frac{\det(C_j)}{\det(C)} \]

**Proof:** The proof is left to Exercise 22.

**Example 3.12:** Use Cramer’s rule to solve each of the following systems.

(a) \[ \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 3 \\ -2 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 11 \end{bmatrix} \]

(b) \[ \begin{bmatrix} 1 & 5 & 0 \\ -2 & -7 & 6 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]

(c) \[ \begin{bmatrix} 1 & 2 & -5 & 1 \\ -4 & -3 & 5 & -4 \\ 2 & 8 & -6 & 1 \\ -1 & -2 & 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 17 \\ 3 \end{bmatrix} \]

(d) \[ \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \]
**Solution:** We do parts (a) and (b) and leave parts (c) and (d) to the student.

For (a), the coefficient matrix

\[
C = \begin{bmatrix}
2 & 4 & 2 \\
0 & 4 & 3 \\
-2 & -3 & 3
\end{bmatrix}
\]

has determinant 34. Hence, \( C \) is invertible and the given system has a unique solution. To find \( x_1 \), we need to compute the determinant of the matrix \( C_1 \) obtained by replacing the first column of \( C \) by the vector \( B = \begin{bmatrix} 6 & 0 & 11 \end{bmatrix}^T \):

\[
\det(C_1) = \begin{vmatrix}
6 & 4 & 2 \\
0 & 4 & 3 \\
11 & -3 & 3
\end{vmatrix} = 170
\]

Thus,

\[
x_1 = \frac{170}{34} = 5
\]

Similarly, we compute \( x_2 \) and \( x_3 \):

\[
x_2 = \frac{\det(C_2)}{\det(C)} = \frac{\begin{vmatrix}
2 & 6 & 2 \\
0 & 0 & 3 \\
-2 & 11 & 3
\end{vmatrix}}{34} = -3
\]

\[
x_3 = \frac{\det(C_3)}{\det(C)} = \frac{\begin{vmatrix}
2 & 4 & 6 \\
0 & 4 & 0 \\
-2 & -3 & 11
\end{vmatrix}}{34} = 4
\]

Thus, the solution is \((5, -3, 4)\). Don’t forget to check it!

For (b), the coefficient matrix \( C \) has determinant 3, so the system has a unique solution. Applying Cramer’s rule, we have:

\[
x_1 = \frac{\det(C_1)}{\det(C)} = \frac{\begin{vmatrix}
1 & 5 & 0 \\
1 & -7 & 6 \\
0 & 3 & -3
\end{vmatrix}}{3} = \frac{18}{3} = 6
\]

\[
x_2 = \frac{\det(C_2)}{\det(C)} = \frac{\begin{vmatrix}
1 & 1 & 0 \\
-2 & 1 & 6 \\
1 & 0 & -3
\end{vmatrix}}{3} = \frac{-3}{3} = -1
\]

\[
x_3 = \frac{\det(C_3)}{\det(C)} = \frac{\begin{vmatrix}
1 & 5 & 1 \\
-2 & -7 & 1 \\
1 & 3 & 0
\end{vmatrix}}{3} = \frac{3}{3} = 1
\]

Thus, the solution is \((6, -1, 1)\).
Exercise Set 3.4

1. For the matrices $A$, $B$, and $D$ of Exercise 1 of Exercise Set 1.3, use the result of Corollary 3.12 to find (if it exists):
   
   (a) $A^{-1}$
   
   (b) $B^{-1}$

   (c) $D^{-1}$

2. For an invertible $n$ by $n$ matrix $C$, prove that:

   (a) If $C$ is upper triangular, then $C^{-1}$ is upper triangular.

   (b) If $C$ is lower triangular, then $C^{-1}$ is lower triangular.

   Hint: Apply Corollary 3.12.

3. Redo Exercise 5 of Exercise Set 1.3 using the result of Corollary 3.12.

4. For an invertible $n$ by $n$ matrix $C$, prove: If $\det(C) = 1$ and all the entries of $C$ are integers, then all the entries of $C^{-1}$ are integers.

5. Let $C$ be the matrix of part (c) of Exercise 3. Use Cramer’s rule to solve the system of linear equations

   $C \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$

6. Consider a system of $n$ linear equations in $n$ variables with coefficient matrix $C$ and constant-term vector $B$. Prove: If $\det(C) = 1$, all the entries of $C$ are integers, and all the entries of $B$ are integers, then the system has a unique solution each of whose entries is an integer.

7. Use cofactor expansion to find the determinant of this matrix:

   $\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{bmatrix}$

8. Given distinct points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $\mathbb{R}^2$, show that an arbitrary point $(x, y)$ is on the line $\overline{PQ}$ if and only if

   $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

9. Let $C$ be the matrix of Exercise 7. Use Cramer’s rule to solve the system of linear equations

   $C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 1 \\ 3 \end{bmatrix}$

10. Given the three noncolinear points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $R = (x_3, y_3, z_3)$ in $\mathbb{R}^3$, show that an arbitrary point $(x, y, z)$ is on the plane determined by $P$, $Q$, and $R$ if and only if

    $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$
11. Redo parts (a) and (b) of Exercise 1 of Exercise Set 2.2 using Cramer’s rule.

12. Given three noncolinear points \( P = (x_1, y_1) \), \( Q = (x_2, y_2) \), and \( R = (x_3, y_3) \) in \( \mathbb{R}^2 \), show that the area of triangle \( PQR \) is equal to the absolute value of

\[
\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}
\]

13. Use the result of Exercise 8 to find the equation of the line in \( \mathbb{R}^2 \) containing the points:

(a) \((-1, -7)\) and \((3, 5)\)  
(b) \((3, -1)\) and \((3, 2)\)

14. Given the four noncoplanar points \((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), \) and \((x_4, y_4, z_4)\) in \( \mathbb{R}^3 \), show that the volume of the tetrahedron having these four points as vertices is equal to the absolute value of

\[
\frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}
\]

15. Use the result of Exercise 10 to find the equation of the plane in \( \mathbb{R}^3 \) containing the points:

(a) \((0, 1, 2), (2, 5), \) and \((3, 5, 2)\)  
(b) \((0, 5, 1), (1, 4, 2), \) and \((4, 1, 3)\)

16. For an \( n \) by \( n \) matrix \( C \), prove that \( C \) is invertible if and only if \( C(\text{adj}(C)) \neq 0 \).

17. Use the result of Exercise 12 to find the area of the triangle in \( \mathbb{R}^2 \) whose vertices are:

(a) \((0, 0), (3, 0), \) and \((3, 4)\)  
(b) \((1, -1), (2, 4), \) and \((3, 1)\)

18. For an \( n \) by \( n \) matrix \( C \), prove that \( \det(\text{adj}(C)) = (\det(C))^{n-1} \).

19. Use the result of Exercise 14 to find the volume of the tetrahedron in \( \mathbb{R}^3 \) whose vertices are:

(a) \((0, 0, 0), (0, 4, 0), (0, 0, 5), \) and \((3, 0, 0)\)  
(b) \((0, 4, 1), (2, 2, 5), (3, 5, 2), \) and \((4, 0, 0)\)

20. For an \( n \) by \( n \) matrix \( C \), prove that \( \text{adj}(\text{adj}(C)) = (\det(C))^{n-2} C \).

21. Prove Theorem 3.9, part 1, in the case \( i = 1 \).

22. Prove Theorem 3.13. Hint: Since \( C \) is invertible, the system \( CX = B \) has the unique solution

\[
X = C^{-1}B = \frac{1}{\det(C)} \text{adj}(C) B
\]
CHAPTER 3 PROBLEMS

1. Compute the determinant of each of the following matrices.
   (a) \[
   \begin{bmatrix}
   1 & 2 \\
   2 & 1
   \end{bmatrix}
   \]
   (b) \[
   \begin{bmatrix}
   2 & -3 & 0 \\
   6 & -2 & 1 \\
   4 & 1 & 2
   \end{bmatrix}
   \]
   (c) \[
   \begin{bmatrix}
   1 & 3 & -5 \\
   1 & 0 & -2 \\
   2 & -3 & -1
   \end{bmatrix}
   \]
   (d) \[
   \begin{bmatrix}
   1 & -1 & 2 & 2 \\
   3 & -2 & 4 & 4 \\
   0 & 1 & -1 & -1 \\
   -2 & 3 & 0 & -3
   \end{bmatrix}
   \]

2. Compute the determinant of each of the following matrices (where \(a, b, c, \ldots\) represent arbitrary numbers).
   (a) \[
   \begin{bmatrix}
   0 & 0 & a \\
   0 & b & c \\
   d & e & f
   \end{bmatrix}
   \]
   (b) \[
   \begin{bmatrix}
   0 & 0 & 0 & a \\
   0 & 0 & b & c \\
   0 & d & e & f \\
   g & h & i & j
   \end{bmatrix}
   \]

3. Solve the following system of equations using Cramer’s rule.
   \[
   \begin{align*}
   2t_1 & - 3t_2 + t_3 = -1 \\
   -2t_1 & + t_2 + 2t_3 = 0 \\
   4t_1 & - t_2 - t_3 = 2
   \end{align*}
   \]

4. Let \(A, B,\) and \(C\) be \(n\) by \(n\) matrices. Suppose that these matrices differ only in row \(r\) and that
   \[
   C(r, j) = A(r, j) + B(r, j)
   \]
   for each \(j, 1 \leq i \leq n.\) Show that
   \[
   \det(C) = \det(A) + \det(B)
   \]

5. Use Cramer’s rule to solve the following system of equations.
   \[
   \begin{align*}
   x & - 4y - 2z = 1 \\
   -2x & + 4y + z = -6 \\
   3x & - 7z = 5
   \end{align*}
   \]

6. Show that \(-1\) and \(3\) are solutions to the equation
   \[
   \begin{vmatrix}
   x^2 & x & 3 & 2x \\
   4 & 3 & 2 & 1 \\
   3 & 1 & 1 & 2 \\
   -1 & 1 & -3 & 2
   \end{vmatrix} = 0
   \]

7. Use Corollary 3.12 to find the inverse of this matrix:

\[
\begin{bmatrix}
1 & 1 & 1 \\
-2 & -1 & 3 \\
3 & 2 & -1
\end{bmatrix}
\]

8. Let \( C \) be an \( n \) by \( n \) matrix each of whose entries is 0 or 1. What is the largest possible value for \( \det(C) \)? Answer for \( n = 1 \), \( n = 2 \), and \( n = 3 \), and then in general, if possible.

9. Use the result of Exercise 4 to express

\[
\begin{vmatrix}
a_1 + b_1 & c_1 + d_1 \\
a_2 + b_2 & c_2 + d_2
\end{vmatrix}
\]

as a sum of four determinants whose entries contain no sums.

10. Given a triangle in \( \mathbb{R}^2 \), let \( \alpha \), \( \beta \), and \( \gamma \) denote the measures of the interior angles and let \( a \), \( b \), and \( c \) denote the lengths of the sides opposite \( \alpha \), \( \beta \), and \( \gamma \), respectively.

   (a) Use trigonometry to show that \( \cos \alpha \), \( \cos \beta \), and \( \cos \gamma \) satisfy the following system of linear equations:

   \[
   \begin{align*}
c \cos \beta & + b \cos \gamma = a \\
c \cos \alpha & + a \cos \gamma = b \\
b \cos \alpha & + a \cos \beta = c
   \end{align*}
   \]

   (b) Use Cramer’s rule to show that

   \[
   \cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}
   \]

11. Use Corollary 3.12 to find the inverse of this matrix:

\[
\begin{bmatrix}
-8 & 4 & -2 & 1 \\
-1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
8 & 4 & 2 & 1
\end{bmatrix}
\]

(This matrix arises in the problem of finding a polynomial \( p(x) \) of degree 3 or less that contains the given points \((-2, p(-2)), (-1, p(-1)), (1, p(1)), \) and \((2, p(2))\).)

12. For an \( n \) by \( n \) matrix \( C \), prove: If \( C \) is invertible, then \( \text{adj}(C) \) is invertible, and

\[
[\text{adj}(C)]^{-1} = \frac{1}{\det(C)} C = \text{adj}(C^{-1})
\]

13. Use Corollary 3.12 to find the inverse of each matrix in Problem 1 (if the matrix is invertible).

14. Let \( A \) be an \( n \) by \( n \) matrix, and define the \( n \) by \( n \) matrix \( B \) by

\[
B(i, j) = A(i, n + 1 - j)
\]

How are \( \det(A) \) and \( \det(B) \) related?

15. Find the inverse of the given matrix using Corollary 3.12.
16. Let $C$ be an $n \times n$ invertible matrix. We ask the following question: How does performing an elementary row operation on a matrix affect its adjoint? In each part, indicate how to obtain the adjoint of the given matrix from the adjoint of $C$. (Does your method work even when $C$ is not invertible?)

(a) $E(1; c, i)C$
(b) $E(2; i_1, i_2)C$
(c) $E(3; c, i_1, i_2)C$

17. Each of the integers 2132, 4641, 7319, and 8294 is a multiple of 13. Use these facts and Cramer’s rule to show that

\[
\begin{vmatrix}
2 & 1 & 3 & 2 \\
4 & 6 & 4 & 1 \\
7 & 3 & 1 & 9 \\
8 & 2 & 9 & 4
\end{vmatrix}
\] is also a multiple of 13.

18. A square matrix $A$ is called idempotent provided $A^2 = A$. Prove: If $A$ is idempotent, then $\det(A) = 0$ or $\det(A) = 1$.

19. Let $F_n$ denote the determinant of the $1, 1, -1$ tridiagonal matrix of order $n$:

\[
F_1 = |1|, \quad F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}, \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}, \quad F_4 = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}, \ldots
\]

Show that $F_1 = 1$, $F_2 = 2$, and then use cofactor expansion to show that $F_n = F_{n-1} + F_{n-2}$. Do you recognize the sequence of numbers $F_1, F_2, F_3, F_4, \ldots$?

20. Let $I$ denote the $n \times n$ identity matrix and let $U$ be an $n \times 1$ matrix such that $U^TU = 1$. The matrix $H = I - 2UU^T$ is called a Householder matrix; see Chapter 2, Problem 20. In each part, give a (nontrivial) example of a Householder matrix of that size, and find its adjoint, determinant, and inverse.

(a) 2 by 2  
(b) 3 by 3  
(c) 4 by 4