The Invertible Matrix Theorem (complete version). Let \( A \) be a square \( n \times n \) matrix. Then the following statements are equivalent. (That is, for a given \( A \), they are either all true or all false.)

a. \( A \) is an invertible matrix.
b. \( A \) is row equivalent to the \( n \times n \) identity matrix.
c. \( A \) has \( n \) pivot positions.
d. The equation \( Ax = 0 \) has only the trivial solution.
e. The columns of \( A \) form a linearly independent set.
f. The linear transformation \( x \mapsto Ax \) is one-to-one.
g. The equation \( Ax = b \) has at least one solution for each \( b \) in \( \mathbb{R}^n \).
h. The columns of \( A \) span \( \mathbb{R}^n \).
i. The linear transformation \( x \mapsto Ax \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \).
j. There is a \( n \times n \) matrix \( C \) such that \( CA = I_n \).
k. There is a \( n \times n \) matrix \( D \) such that \( AD = I_n \).
l. \( A^T \) is an invertible matrix.

* m. The columns of \( A \) form a basis for \( \mathbb{R}^n \).

* n. \( \text{Col} \ A = \mathbb{R}^n \).

* o. \( \text{dim Col} \ A = n \).

* p. \( \text{rank} \ A = n \).

* q. \( \text{Nul} \ A = \{0\} \).

* r. \( \text{dim Nul} \ A = 0 \).

* s. The number 0 is not an eigenvalue of \( A \).

* t. The determinant of \( A \) is not zero.

We could also add a bunch of statements about \( \text{Row} \ A \) and \( A^T \), but that’s too much writing for stuff that’s so obvious.

* Characterizations added since the original statement of the IMT in Chapter 2.