Review Exercises:

1.

2.

3.

4. Mark each statement True or False. *Justify your answers, i.e., explain.* Try to do these without looking in the text (at least at first), since you won’t be able to use the text during the exam!

(a) If $A$ is row equivalent to the identity matrix $I$, then $A$ is diagonalizable.

   *False.* If $A$ is row equivalent to the identity matrix, then $A$ is invertible. We gave an example in 5.3 #31 of a matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ that is invertible but not diagonalizable.

(b) Similar matrices always have exactly the same eigenvalues.

   *True.* This is Theorem 5.4. If $B = P^{-1}AP$, then you can check $B - \lambda I = P^{-1}(A - \lambda I)P$. It follows that $\det(B - \lambda I) = \det(P^{-1})\det(A - \lambda I)\det P$. The righthand side is a product of real numbers that can be rearranged so that $\det(P^{-1}) = \det(P)^{-1}$ and $\det P$ are next to each other and cancel. Thus $\det(B - \lambda I) = \det(A - \lambda I)$, *i.e.*, $A$ and $B$ have the same characteristic polynomial. Therefore they have the same eigenvalues as well. (See the proof on p. 315 for more detail.)

(c) If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.

   *True.* $AB$ and $BA$ are similar if there is an invertible matrix $P$ so that $P^{-1}BAP = AB$. Let $P = B$ to get $B^{-1}BAB = IAB = AB$.

(d) If a $5 \times 5$ matrix $A$ has fewer than 5 distinct eigenvalues, then $A$ is not diagonalizable.

   *False.* Let $A = I_5$. Being a diagonal matrix, it’s clearly diagonalizable, yet its only eigenvalue is 1.

5. (4.2) p. 234 #12

   $W$ is *not* a subspace of $\mathbb{R}^4$ because $W$ doesn’t contain the zero vector of $\mathbb{R}^4$. If the fourth entry of a vector in $W$ is 0, then $d = 0$, and then the third entry must be 1. ($W$ also fails the other two subspace conditions.)

6. (4.3) p. 243 #8

   This set is *not* a basis for $\mathbb{R}^3$. There are more than three vectors (more than the number of entries in each vector), so the set cannot be linearly independent. To see if it spans form a $3 \times 4$ matrix from these vectors and reduce to echelon form. This will reveal that every row has a pivot position, so the columns do span $\mathbb{R}^3$ by Theorem 1.4.
7. (4.3) p. 243 #14

For Nul A, put B in reduced echelon form and then, from this, get the parametric vector form of the solution set to Ax = 0. The two vectors that appear in that description give the basis \[
\begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
-4 \\
0 \\
7/5 \\
1 \\
0
\end{bmatrix}
\] for Nul A.

The pivot columns of A give a basis for Col A, and from B we can see that columns 1, 3, and 5 are pivot columns. Thus \[
\begin{bmatrix}
1 \\
2 \\
1 \\
1 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
-5 \\
-5 \\
0 \\
5 \\
-2
\end{bmatrix}
\] is a basis for Col A.

8. (4.5) p. 261 #26

Choose n vectors \(v_1, \ldots, v_n \in H\) so that \(\{v_1, \ldots, v_n\}\) is a basis for H. Then \(\{v_1, \ldots, v_n\}\) is a linearly independent set of vectors in V. Therefore, since V is n-dimensional, \(\{v_1, \ldots, v_n\}\) is automatically a basis for V by The Basis Theorem. Thus we have \(H = \Span\{v_1, \ldots, v_n\} = V\).

9. (4.6) p. 269 #4

\(\text{rank } A = 3\) (B has 3 nonzero rows)

\(\text{rank } A + \dim \text{Nul } A = n \Rightarrow 3 + \dim \text{Nul } A = 6 \Rightarrow \dim \text{Nul } A = 3\).

\(\text{Basis for } \text{Col } A: \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 10 \\ -5 \end{bmatrix} \right\} \) (pivot columns of A)

\(\text{Basis for } \text{Row } A: \quad \{ (1, 1, -3, 7, 9, -9), (0, 1, -1, 3, 4, -3), (0, 0, 0, 1, -1, -2) \} \) (nonzero rows of B)

\(\text{Basis for } \text{Nul } A: \quad \text{Reduce } B \text{ to rref to get the solution set of } Ax = 0 \text{ in parametric vector form. The vectors in that description give the basis } \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\
-9 \\ -7 \\ 1 \\ 1 \\ 0 \\ 0 \\
-2 \\ -3 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.\)

10. (4.6) p. 269 #10

\(\text{rank } A + \dim \text{Nul } A = n \Rightarrow \text{rank } A + 5 = 6 \Rightarrow \text{rank } A = 1 \Rightarrow \dim \text{Col } A = 1\).

11. (4.6) p. 269 #16

\(\text{rank}(A) \leq 4 \text{ because } A \text{ has 4 columns. So } \text{rank } A + \dim \text{Nul } A = n \Rightarrow \dim \text{Nul } A = n - \text{rank } A \geq 4 - 4 = 0. \text{ The smallest possible dimension for } \text{Nul } A \text{ is 0.}\)
12. (4.6) p. 270 #20

The corresponding coefficient matrix $A$ is $6 \times 8$. Since there are two free variables, the solution set of the corresponding homogeneous system will be 2-dimensional, i.e. $\dim \text{Nul } A = 2$. Then $\text{rank } A = 8 - 2 = 6$, so that each of the 6 rows of $A$ has a pivot position. By Theorem 1.4, $Ax = b$ has a solution for every $b \in \mathbb{R}^6$, so the system cannot be made inconsistent by changing the constants.

13. (5.1) p. 308 #16

Describe the solution set of $(A - 4I)x = 0$ for $x$ in parametric vector form. The vectors appearing there give the basis \[
\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

14. (5.1) p. 309 #32

An eigenvalue is 1. This is because the axis of rotation is a 1-dimensional subspace that’s invariant under the rotation. In fact, each vector $x$ on this line is fixed: $Ax = x$, i.e. $Ax = 1x$. So the axis of rotation is the eigenspace corresponding to $\lambda = 1$.

15. (5.2) p. 317 #14

The characteristic polynomial is $P_A(\lambda) = \det(A - \lambda I) = (1 - \lambda)(\lambda^2 - 3\lambda - 28) = (1 - \lambda)(\lambda + 4)(\lambda - 7)$

16. (5.3) p. 326 #18

Denote the given matrix by $A$. We are given that $\lambda = 5$ is one eigenvalue. The easiest way to find another eigenvalue is to multiply the given eigenvector $(-2, 1, 2)$ by $A$ to get $(6, -3, -6)$. Thus $\lambda = -3$ is an eigenvalue. There could be a third eigenvalue, but we’ll see that in fact the above two are the only ones.

For the eigenspace corresponding to $\lambda = 5$, put the solution set of $(A - 5I)x = 0$ in parametric vector form to obtain the basis \[
\begin{pmatrix} -4/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix}.
\]
(Since this eigenspace is two-dimensional, $\lambda = 5$ must be an eigenvalue of multiplicity 2. Thus, counting with multiplicity, we already have 3 eigenvalues, so there won’t be any more.) For the eigenspace corresponding to $\lambda = -3$, put the solution set of $(A + 3I)x = 0$ in parametric vector form to obtain the basis \[
\begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix}.
\]

Now form the invertible matrix $P = \begin{bmatrix} -4/3 & 1/3 & -1 \\ 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$ from the eigenvectors and the diagonal matrix $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ from the eigenvalues. Then $A = PDP^{-1}$.

[Note: You can switch the columns of $P$ around and obtain another valid matrix, as long as you also switch the columns of $D$ around in the same way.]
17. (5.3) p. 326 #26

Yes, it’s possible that $A$ is not diagonalizable. By Theorem 5.7, $A$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces is 7. There are three eigenspaces, and we know two of them have dimensions 2 and 3, respectively. The sum of the dimensions is 5 so far. The third eigenspace might only be one-dimensional, giving a total of 6. If so, $A$ is not diagonalizable. (But it is if the last eigenspace is two-dimensional.) [Note that an eigenspace always has at least dimension 1. Why?]

18. (5.3) Actually diagonalize the matrix in #5 on the text’s Third Practice Exam

Since $A$ is triangular, its eigenvalues are $\lambda = 3$, $\lambda = -5$, and $\lambda = 2$, and bases for the corresponding eigenspaces are \[
\begin{align*}
\text{eigenspace for } \lambda = 3: & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
\text{eigenspace for } \lambda = -5: & \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \\
\text{eigenspace for } \lambda = 2: & \begin{bmatrix} 4/7 \\ 3/7 \\ 1 \end{bmatrix}.
\end{align*}
\]
Thus we can take $P = \begin{bmatrix} 1 & -1 & 4/7 \\ 0 & 1 & 3/7 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. (Again columns can be switched as long as it’s done to both.) Then $A = PDP^{-1}$.

19. (5.3) Show that if $A$ and $B$ are similar, then $\det A = \det B$.

Let $P$ be an invertible matrix such that $B = P^{-1}AP$. Then $\det B = \det(P^{-1}AP) = \det(P^{-1}) \det A \det P = \det(P)^{-1} \det P \det A = \det A$. 